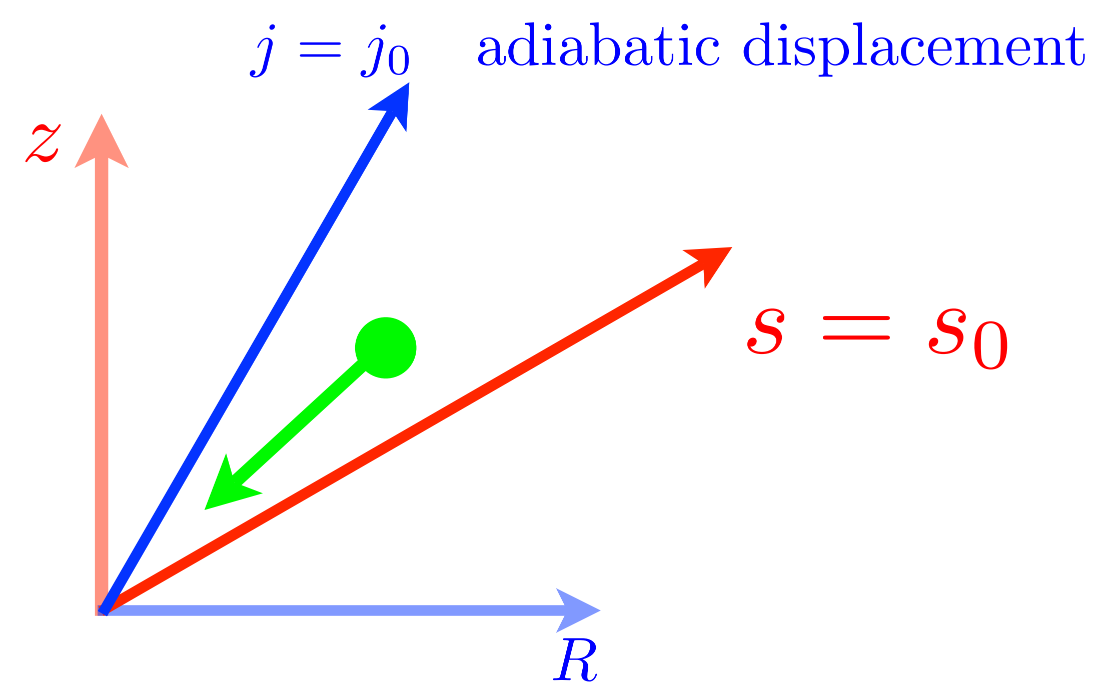
Instabilities in Protoplanetary Disks

- Compare lines of constant entropy with lines of constant angular momentum assuming adiabatic displacement.
    - Displaced parcels move vertically towards the line of constant entropy
    - Displaced parcels move radially towards the line of constant angular momentum.



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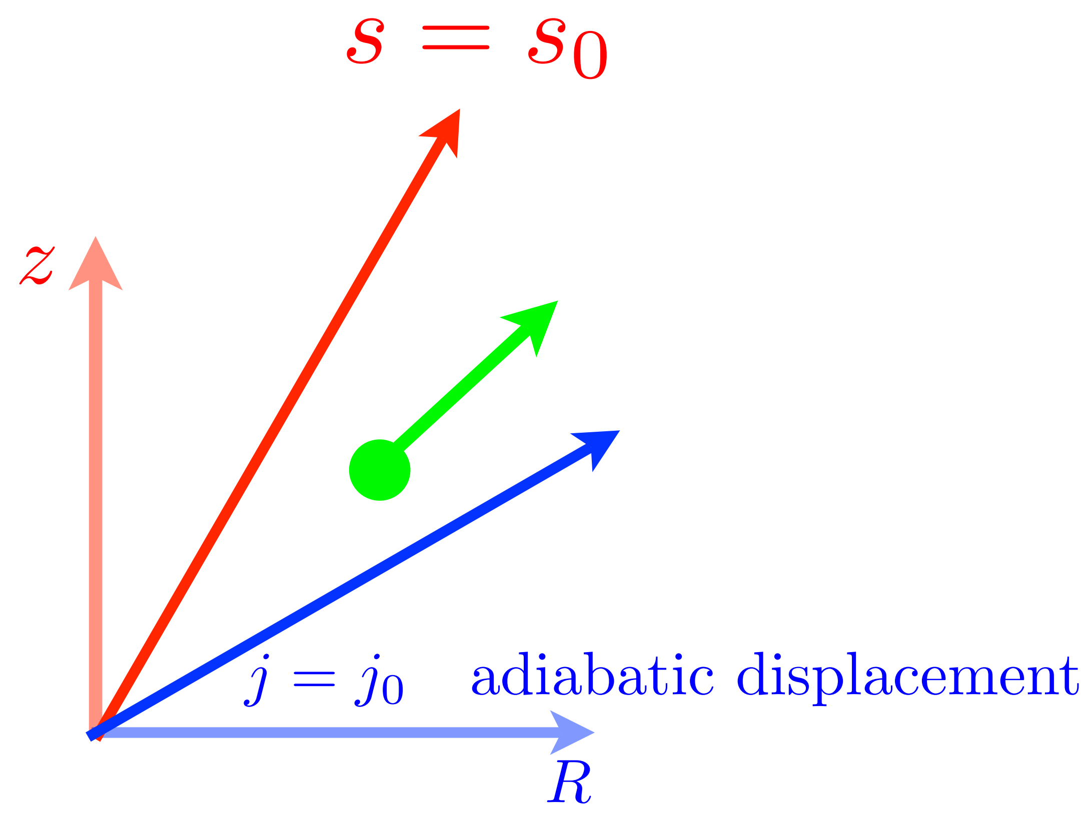
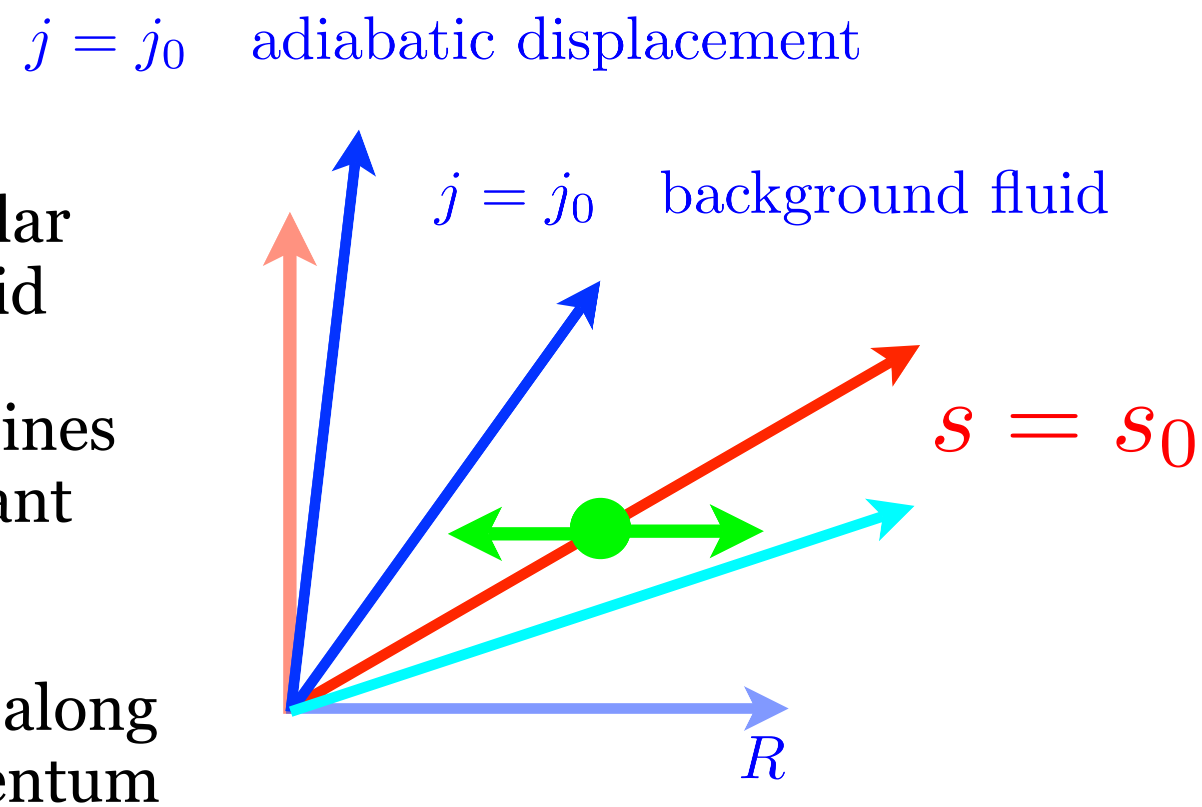


Adding cooling

- Adding cooling requires us to also consider the line of constant angular momentum of the background fluid
  - ➡ Constant angular momentum lines can't straddle the line of constant entropy
- Rapid cooling: nearly free motion along the line of constant angular momentum for the background fluid



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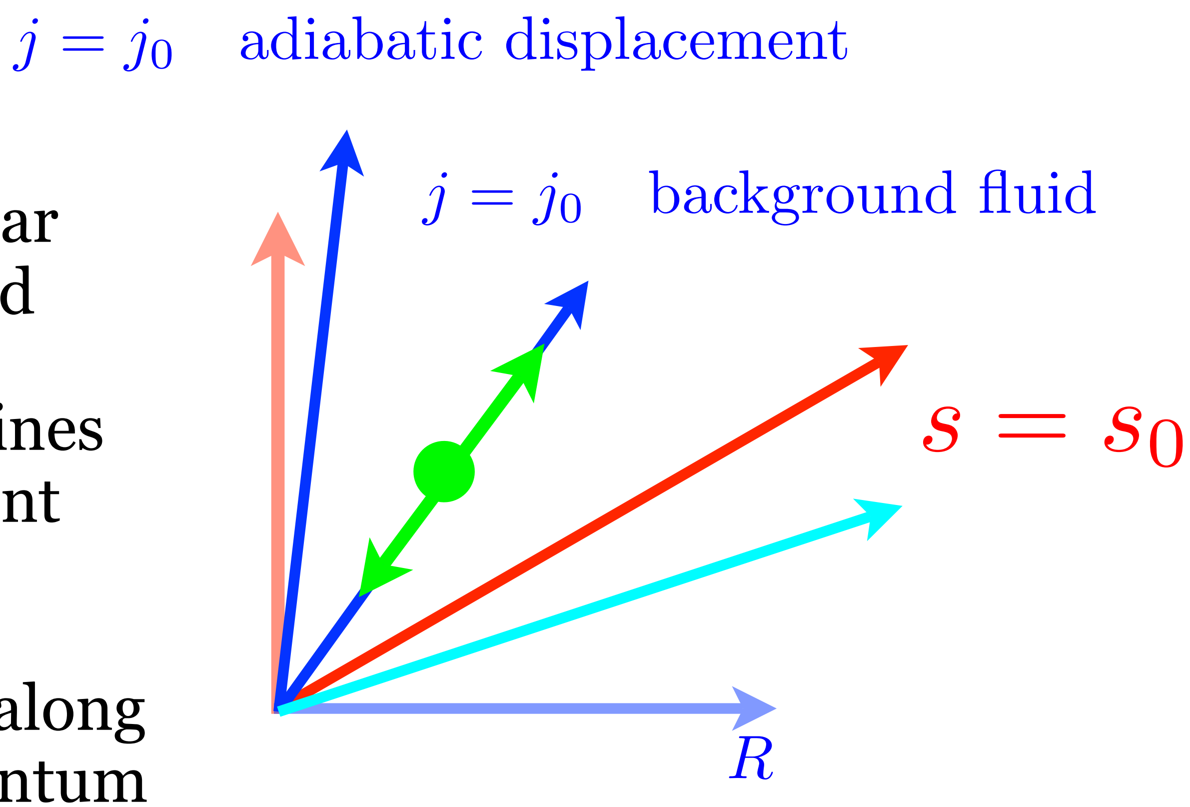

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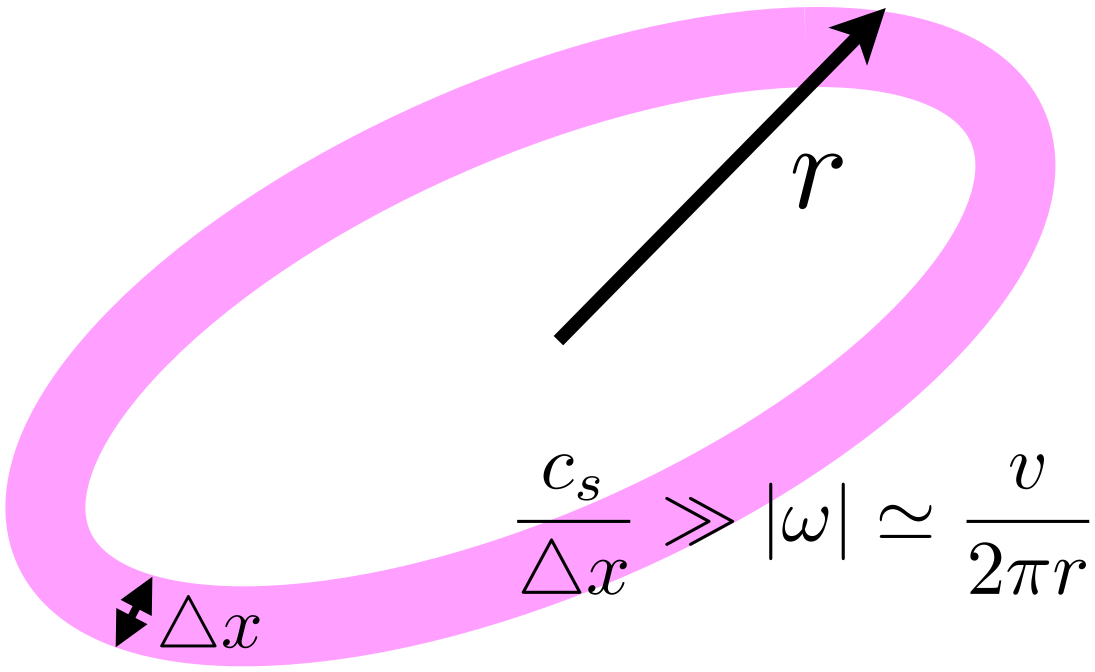
Overstability

- Adiabatic system: all forces are conservative
  - ➡ No possibility of an overstability
- Non-adiabatic motion: room to play
  - ➡ Lesur & Papaloizou 2010
- Numerics
  - ➡ Klahr, Nelson et al
- Here: zero perturbed pressure
  - ➡ Parcel analysis

$$-\frac{1}{\rho}\boldsymbol{\nabla} p \propto \boldsymbol{\nabla} p^{\frac{\gamma-1}{\gamma}}$$



$$\frac{c_s}{\triangle x} \gg |\omega| \simeq \frac{v}{2\pi r}$$

Dispersion Relation

- Linearized equations (assuming no pressure perturbations)

$$\frac{\partial u_R}{\partial t} = -\kappa_R^2 R' - \kappa_z^2 z' - \frac{s'}{\gamma}\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial R},$$
$$\frac{\partial u_z}{\partial t} = -\frac{s'}{\gamma}\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z},$$
$$\frac{\partial s'}{\partial t} = -\frac{s'}{\tau_c} - u_R\frac{\partial \overline{s}}{\partial R} - u_z\frac{\partial \overline{s}}{\partial z},$$

- Dispersion relation:
  - Radial epicyclic
  - Vertical Brunt-Vaisala
  - Perturbative term

$$\kappa_R^2 \equiv R_0^{-3}\partial_R \overline{j}^2$$
$$\kappa_z^2 \equiv R_0^{-3}\partial_z \overline{j}^2$$

$$\left(\omega^2 - \kappa_R^2\right)\left(\omega^2 + \frac{i\omega}{\tau_c} - N_z^2\right) - \frac{\partial \overline{s}}{\partial R}\frac{A_z\kappa_z^2}{\gamma} - N_R^2\omega^2 = 0$$

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Overstability Criterion

$$\omega = \pm\kappa_R + \delta_\pm$$

- Perturb around the radial epicyclic solution

$$\mathrm{Im}\,\delta = -\frac{\frac{\partial \overline{s}}{\partial R}}{2\kappa_R}\frac{\left(A_z\kappa_z^2/\gamma + A_R\kappa_R^2/\gamma\right)\left(\frac{\kappa_R}{\tau_c}\right)}{\left(\kappa_R^2 - N_R^2 - N_z^2\right)^2 + \left(\frac{\kappa_R}{\tau_c}\right)^2}.$$

- Positive imaginary component: growth

- Instability criterion

$$\partial_R\overline{s}\left(A_z\kappa_z^2 + A_R\kappa_R^2\right) < 0$$

Isothermal Disk

- Even if the radial entroy gradient is postive at the midplane, it will be negative at altitude

$$\frac{\partial \overline{s}}{\partial R} = \left\{ -q + (\gamma - 1) \left[p - \left(\frac{3}{2} - \frac{q}{2} \right) \frac{z^2}{H^2} \right] \right\} \frac{1}{R}$$

$$A_z \kappa_z^2 + A_R \kappa_R^2 = \frac{H^2 \Omega_K^4}{R} \left[p + q - \left(\frac{3}{2} + \frac{q}{2} \right) \frac{z^2}{H^2} \right]$$

- But the radial pressure force also changes direction

$$\frac{z_s}{H} = \left[2 \left(\frac{p - \frac{q}{\gamma - 1}}{3 - q} \right) \right]^{1/2}$$

$$\frac{z_p}{H} = \left[2 \left(\frac{p + q}{3 + q} \right) \right]^{1/2}$$

- Slow growth rates:
  - ➡ 100 orbits or more