Stability of structural features and vortices in protoplanetary disks

Topics:

Stability of edges and ring-like features.

Stability of vortices - parametric/elliptic instability

Baroclinic instability and vortices

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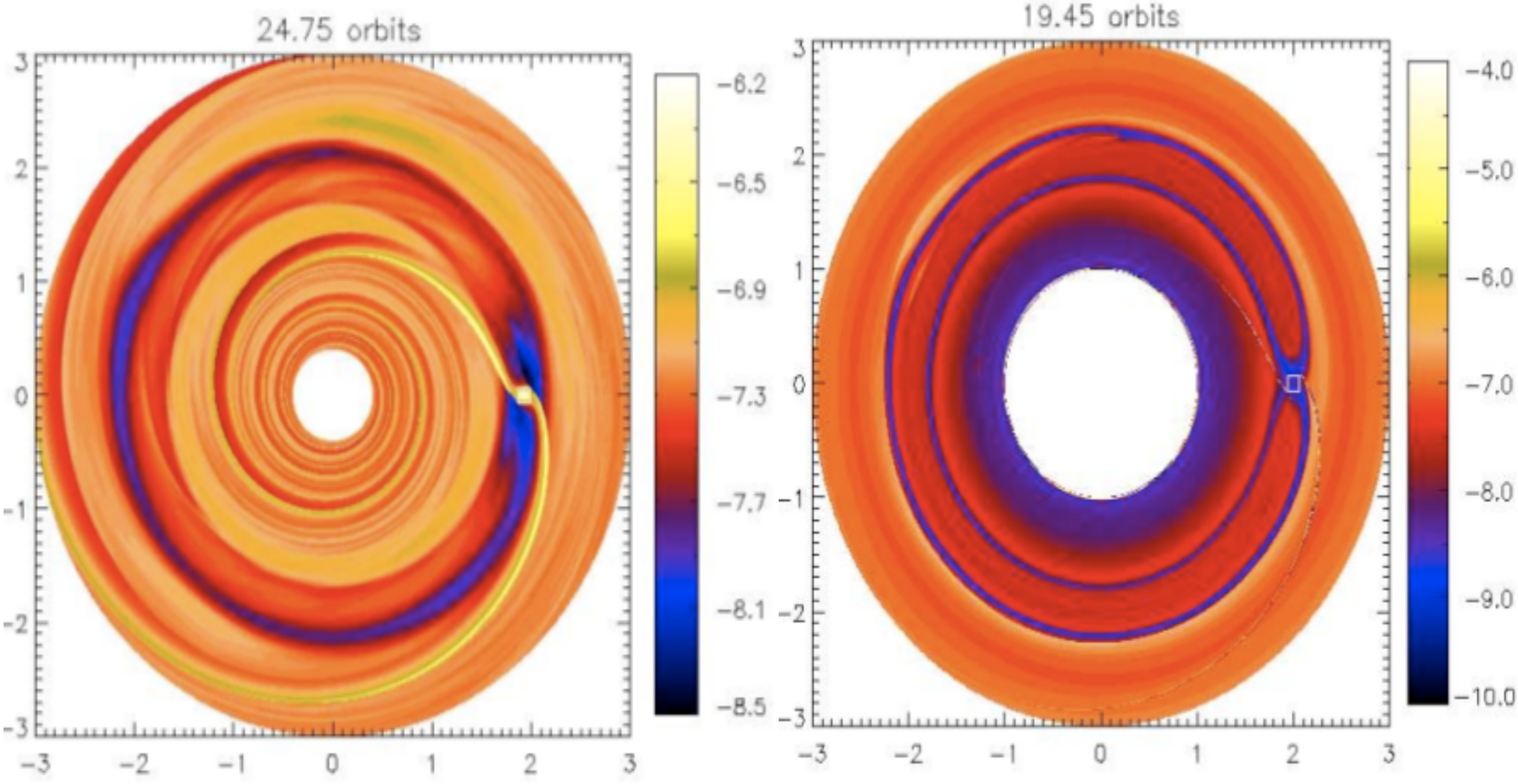
Structural features that can be unstable are:

1) Regions associated with pressure extrema – eg. material near interface between a live (MRI active) and dead zone, Lyra & Maclow (2011)

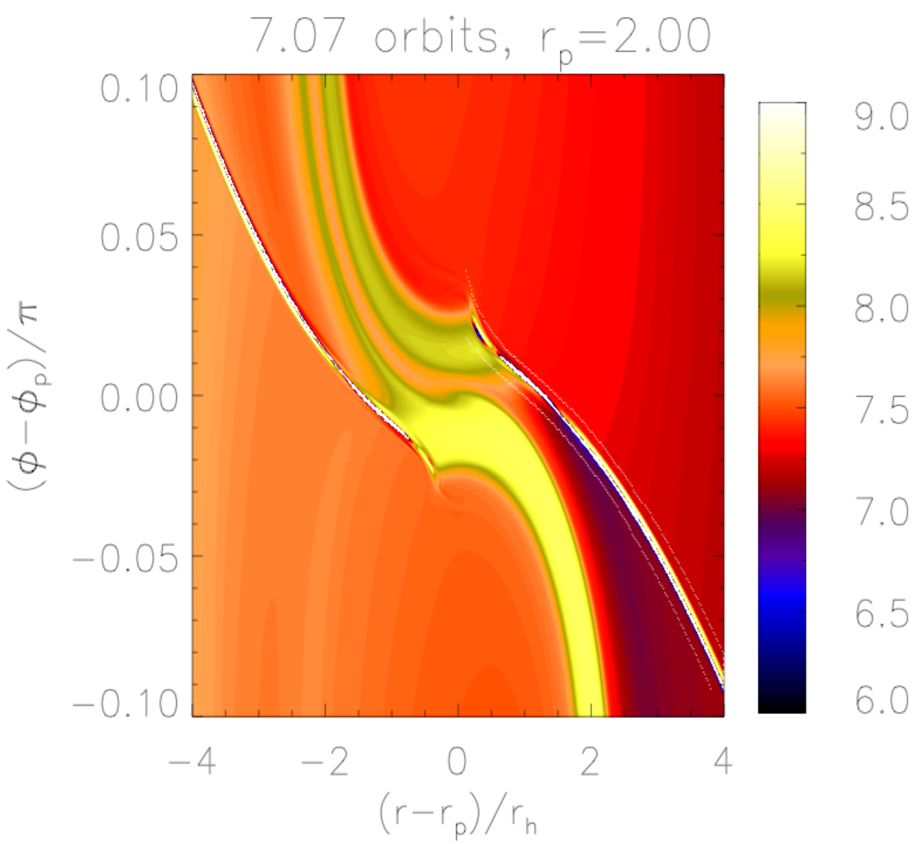
2) Disk edges…. produced by orbiting protoplanets can also undergo an instability leading to large scale vortex production (eg. Koller et al 2003, De Val-borro et al 2006)

Here we focus on 2). However, discussion of 1) Is similar

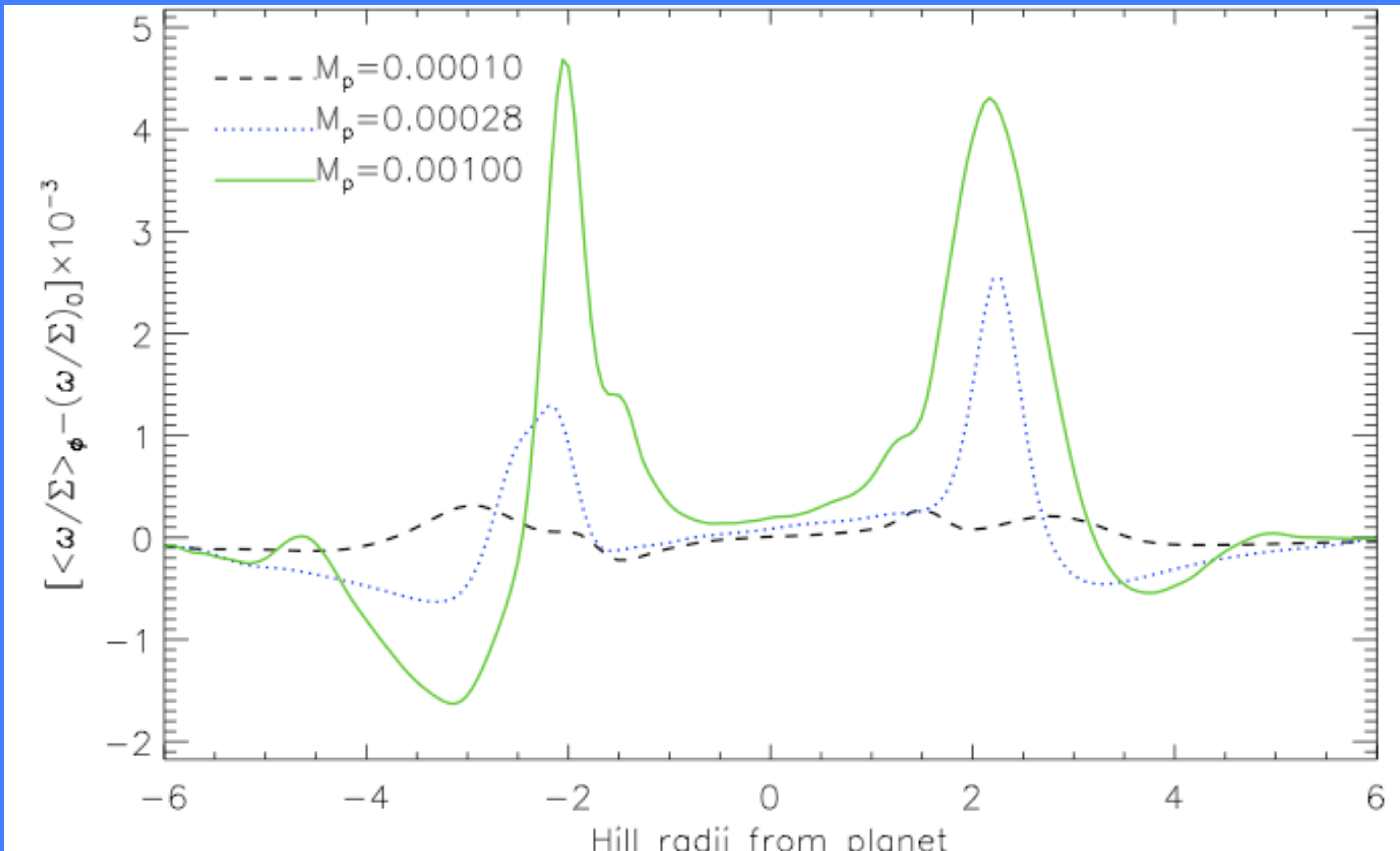
Surface density and vortensity distributions set up by a Saturn mass protoplanet in a locally isothermal disk (H/r =0.05). We consider 2D barotropic case. Instability is expected to be 2D (see Lin 2012, Meheut et al. 2012).



Log vortensity (log ω/Σ) changes induced by shocks produced by a Saturn mass protoplanet in an inviscid disc. The broad high level regions originate in the coorbital region



Induced vortensity perturbation for different planet masses showing inner and outer localized extrema (potential sites of instability)



Stability of

slender surface density /vortensity rings in 2D barotropic disks. Papaloizou & Pringle (1985) derived governing equation in 3D. Vertical motions not important so vertical averaging is possible

The equation governing linear perturbations is

$$\frac{d}{dr}\left(\frac{c^2 r\Sigma}{\kappa^2-\bar{\sigma}^2}\frac{dW}{dr}\right)+\left\{\frac{m}{\bar{\sigma}}\frac{d}{dr}\left[\frac{c^2\kappa^2}{\eta(\kappa^2-\bar{\sigma}^2)}\right]-r\Sigma-\frac{m^2c^2\Sigma}{r(\kappa^2-\bar{\sigma}^2)}\right\}W=0.$$

where $W \equiv \Sigma'/\Sigma$ is the fractional density perturbation, η is the vortensity, κ is the epicyclic frequency and $\bar{\sigma} \equiv \sigma + m\Omega(r)$ is the Doppler-shifted frequency.

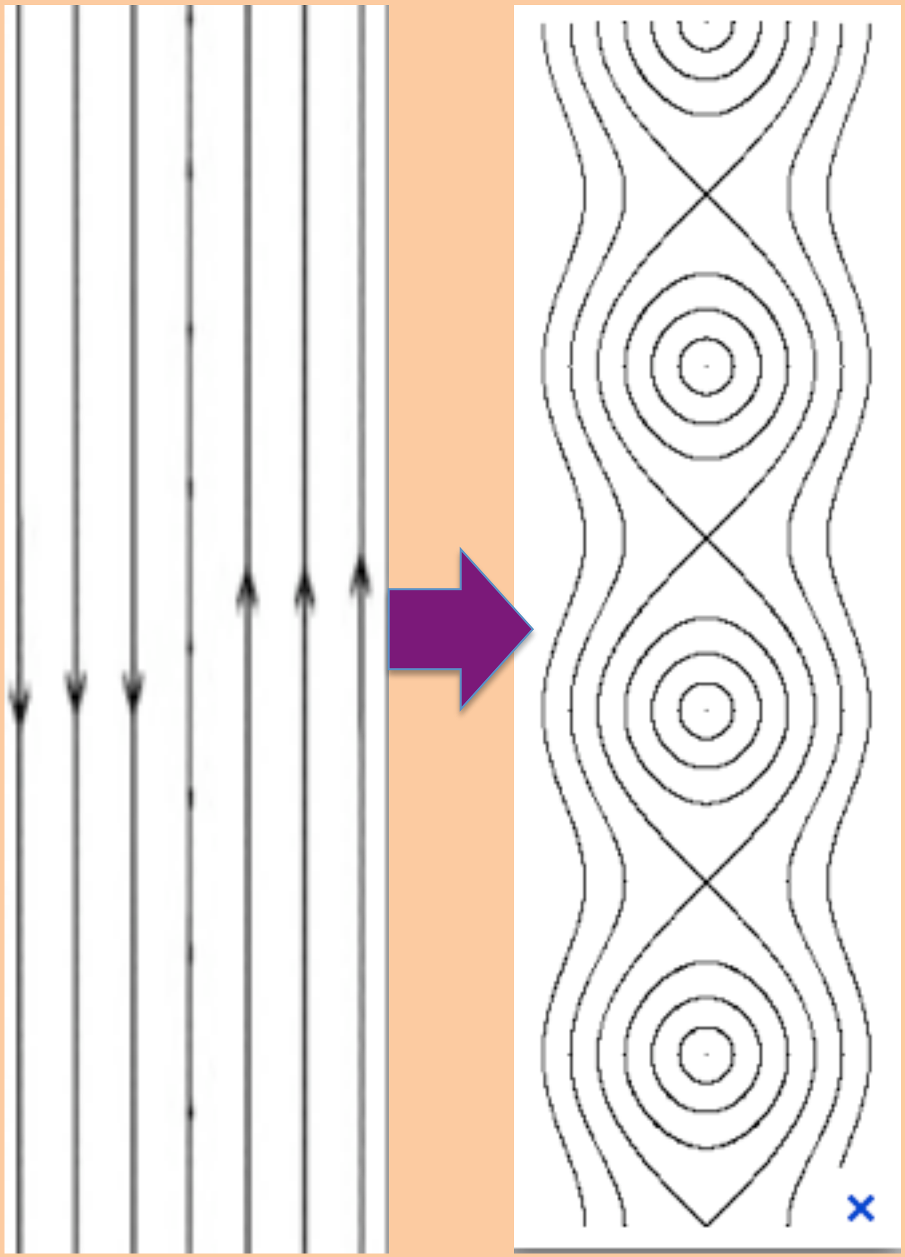
We consider 'co-rotational' modes for which $\kappa^2 \gg |\bar{\sigma}^2|$ and a ssume a thin disk for which we have $c/(r\Omega) \ll 1$. In this limit, which applies for $m \ll r/H$, we have

$$\frac{d}{dr}\left(\frac{rc^2\Sigma}{\kappa^2}\frac{dW}{dr}\right)+\left\{\frac{m}{\bar{\sigma}}\frac{d}{dr}\left[\frac{c^2}{\eta}\right]-r\Sigma\right\}W=0,$$

Vortensity minima (often associated with density maxima) may be associated with unstable modes with corotation point at the minimum.

Condition for instability:

Conservation of specific vorticity
ω/Σ
on streamlines in 2D requires this to have an extremum for a linear instability



Analogy with Kelvin Helmholtz instability

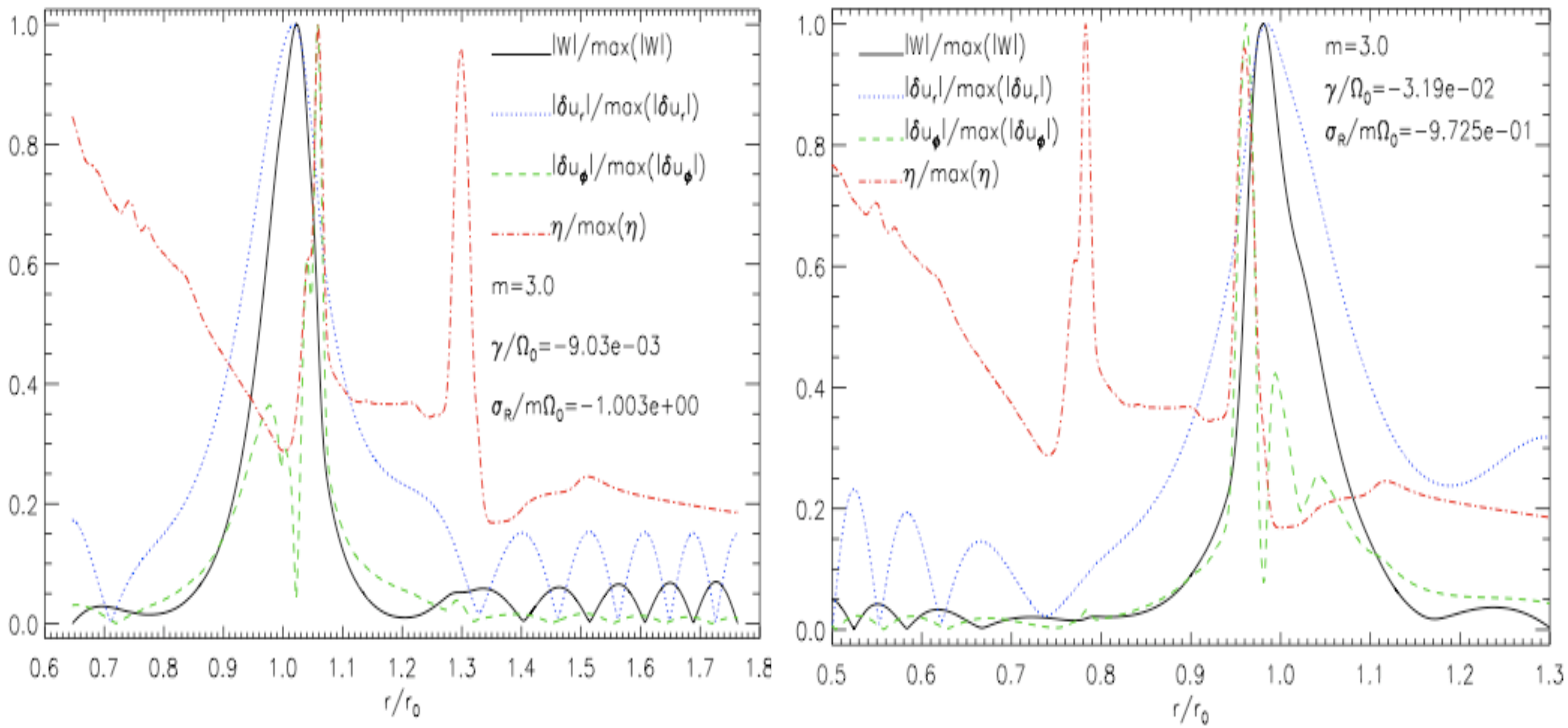
Governing eq. for velocity perturbation is

$$f'' - \left(k^2 + \frac{u''}{(u-c)}\right) f = 0$$

compared to

$$\frac{d}{dr}\left(\frac{r c_s^2 \Sigma}{\kappa^2}\frac{dW}{dr}\right) + \left\{\frac{m}{\bar{\sigma}}\frac{d}{dr}\left[\frac{c_s^2}{\eta}\right] - r\Sigma\right\} W = 0$$

Typical growth rate, $\gamma \sim m\Delta\Omega$ (shear rate)

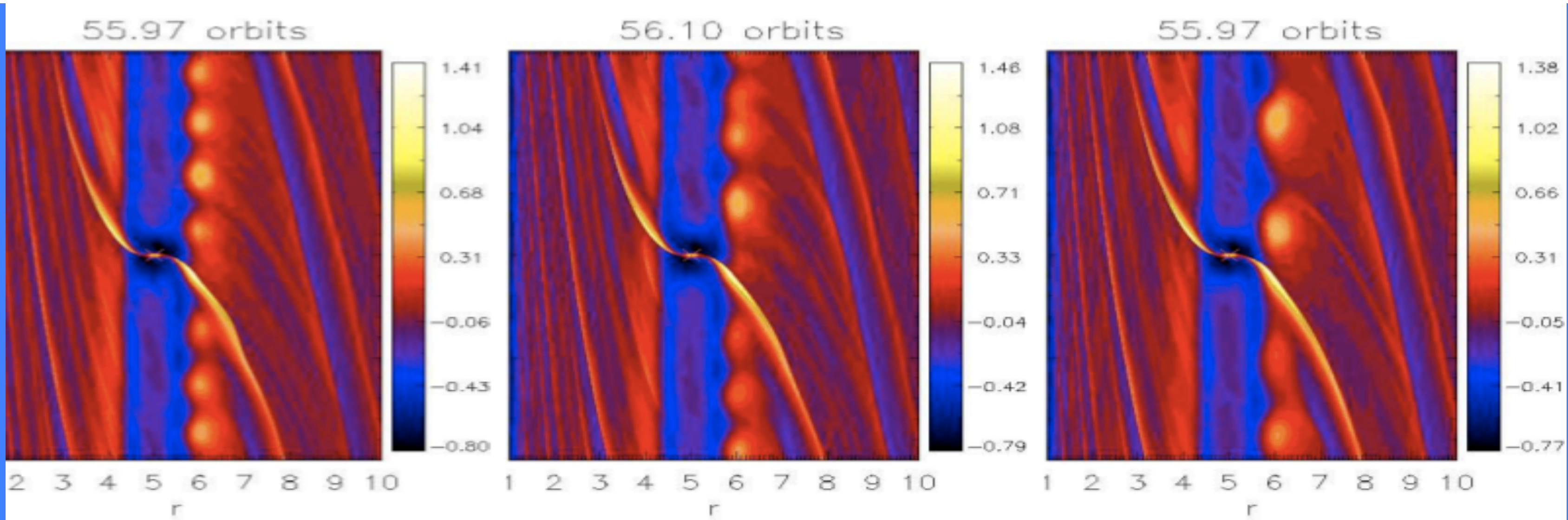


Unstable modes with m=3 associated with inner and outer rings: Modes with low m are localised near surface density maxima and decay away exponentially with some weak wave emission.

Efect of Self-gravity

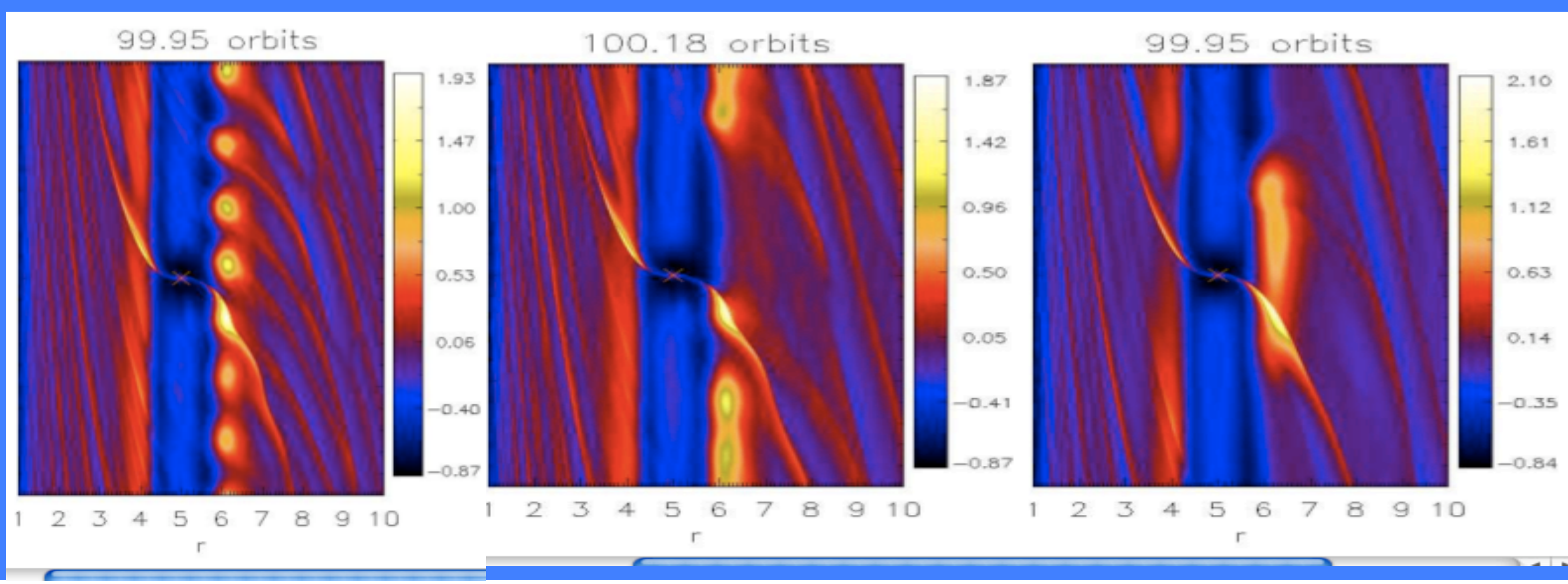
The strength of this is measured by when
Toomre's $Q = \Omega c / \pi G \Sigma$,
which has to exceed unity to be stable to
axisymmetric modes,
attains a value approaching unity.

Vortices associated with vortensity minima then
attain smaller azimuthal scales, being eventually
replaced by large scale self-gravitating edge
modes associated with vortensity maxima.



Surface density for Q =3,4,8: Top, linear phase,

Botom, in the nonlinear regime.



Unstable modes peaked at outer vortensity maxima

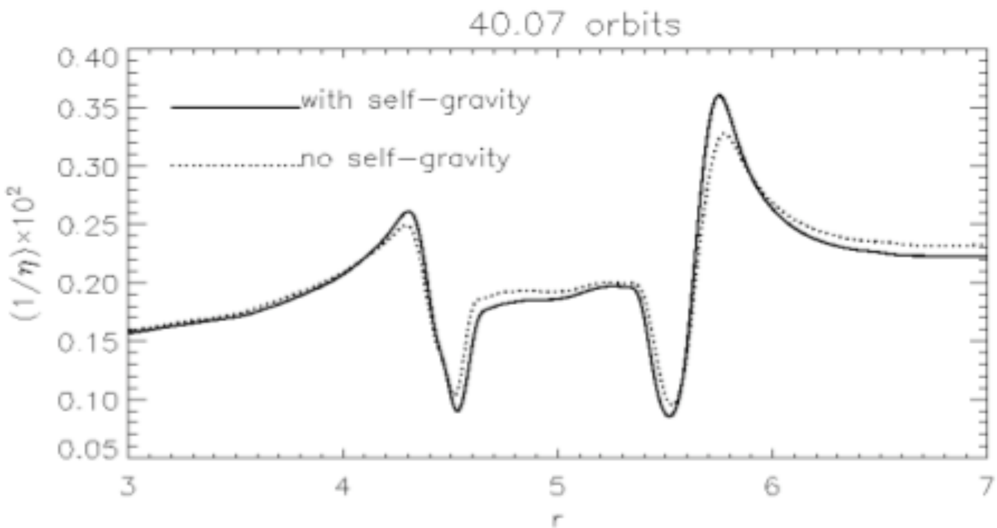
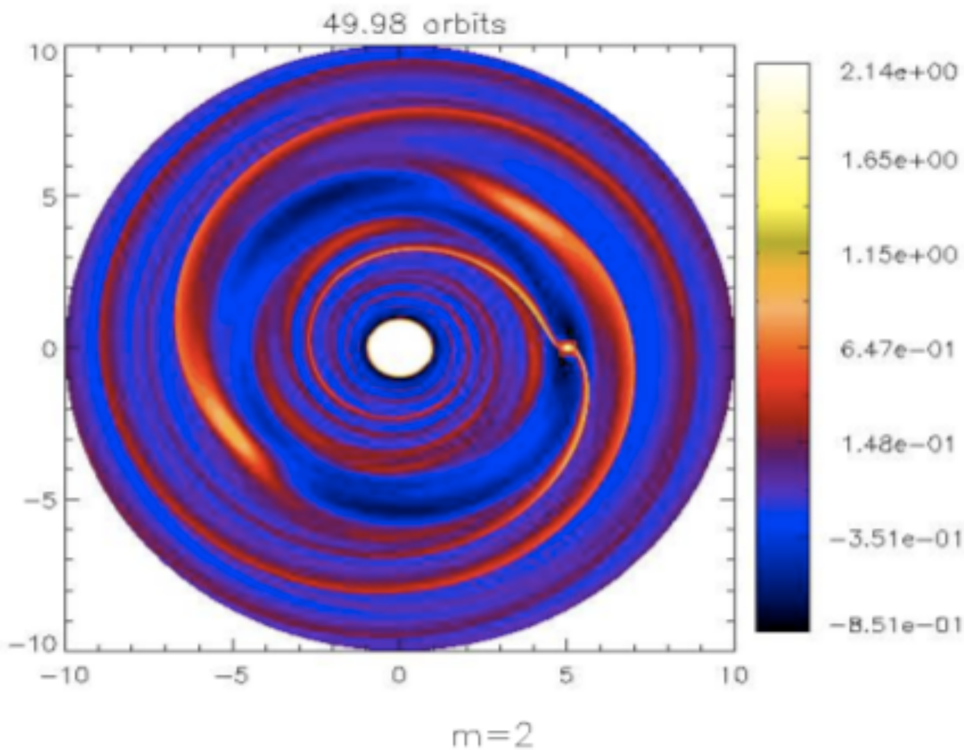


Figure 1. Gap profiles opened by a Saturn-mass planet in an initial disc model with $Q_o = 4$. The inverse vortensity η^{-1}, obtained with (solid) and without (dotted) self-gravity included is



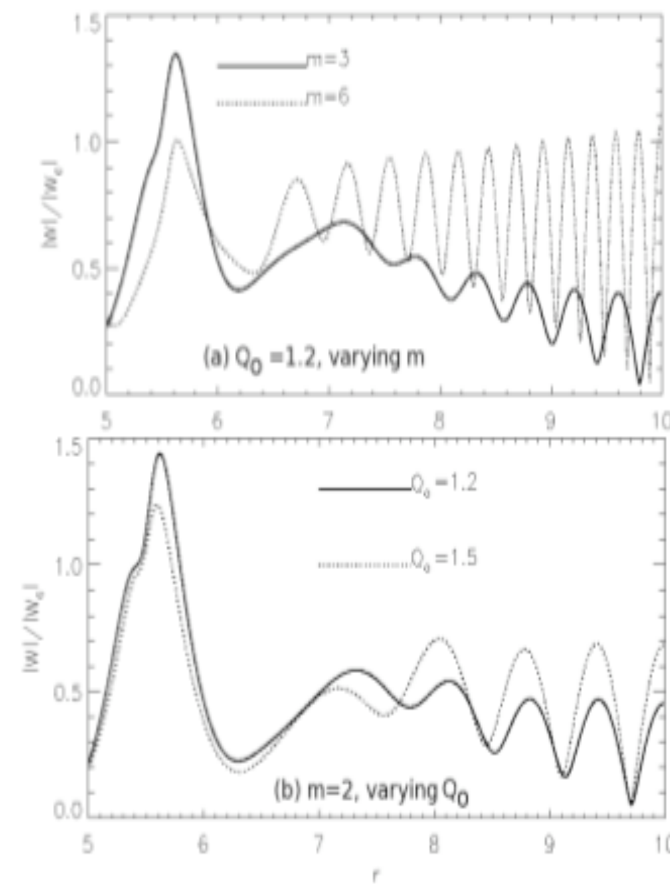


Figure 8. The effect of azimuthal wave-number m (a) and the effect of disc mass, parametrised by Q_o (b) on linear edge modes.

Stability Issues

Conservation of specific vorticity enables long term survival of vortices produced by instabilities in 2D.

However, vortices are subject to a generic form of instability In 3D…

Expect elliptical instability…. Parametric excitation of inertial/gravity waves..

see eg. Lesur & Papaloizou (2009)

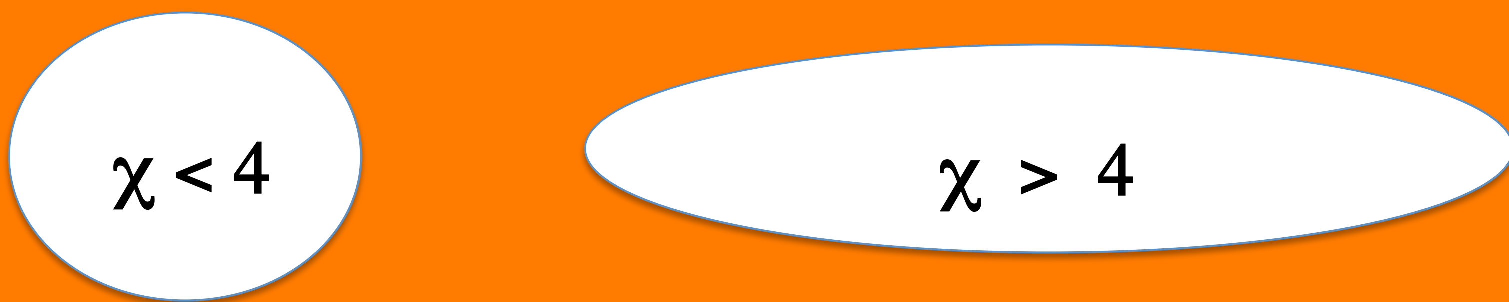
Kida vortex: Analytic solution for constant 2D (negative) vorticity patch with elliptical cross section in incompressible limit (scale << H):

Flow inside the core has a simple linear form

v_x α y, and v_x α x, vorticity is related to the aspect ratio

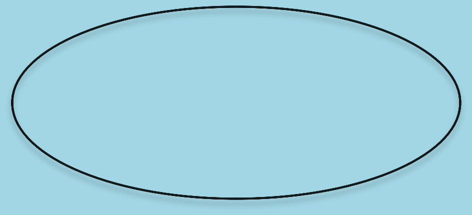
χ = a/b by $2\omega/(3\Omega) = -\ (\chi + 1)/\ (\chi\ (\chi - 1))$...

Only for χ > ~ 4 (weak vortex) is there a pressure maximum.



Can also find local solutions with non uniform vorticity distributions and with non uniform density corresponding to a dust component (see A. Railton)

Instability of non circular streamlines (elliptic instability)



Neglect vertical stratification of background - Disk supports inertial and gravity waves asociated with radial stratification

Dispersion relation for inertial modes iin the radially stratified case is

$$\sigma^2 = \frac{(\Omega^2 + N^2)k_z^2}{|\mathbf{k}|^2}$$

As $|k| \longrightarrow$ infinity group velocity $\longrightarrow 0$ local disturbances can stay localized on streamlines.
Component of k perpendicular to u is large.
Can obtain an equation governing disturbances localized On streamlines

Equation governing disturbances localized on streamlines

$$\frac{D^2}{Dt^2}\left(\boldsymbol{\xi}+\frac{\mathbf{a}(\boldsymbol{\xi}\cdot\mathbf{a})}{k_z^2}\right)+2\boldsymbol{\Omega}\times\frac{D\boldsymbol{\xi}}{Dt}-\frac{2}{k_z^2}\mathbf{u}\cdot\nabla\mathbf{a}\frac{D(\boldsymbol{\xi}\cdot\mathbf{a})}{Dt}=$$

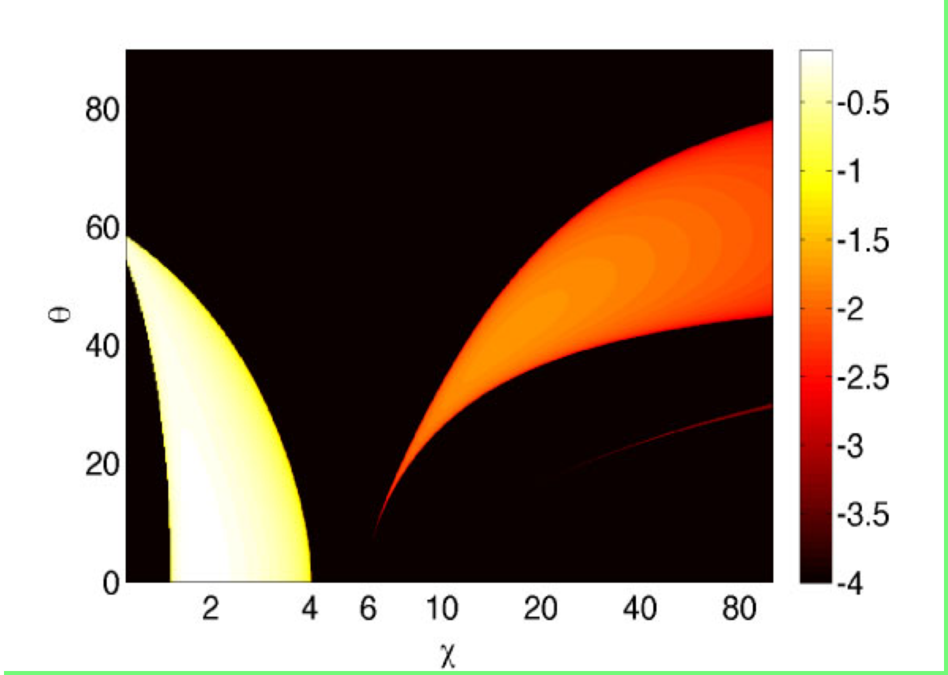
$$\mathbf{C}(\boldsymbol{\xi})+\frac{\boldsymbol{\xi}\cdot\mathbf{a}}{k_z^2}\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{a})$$

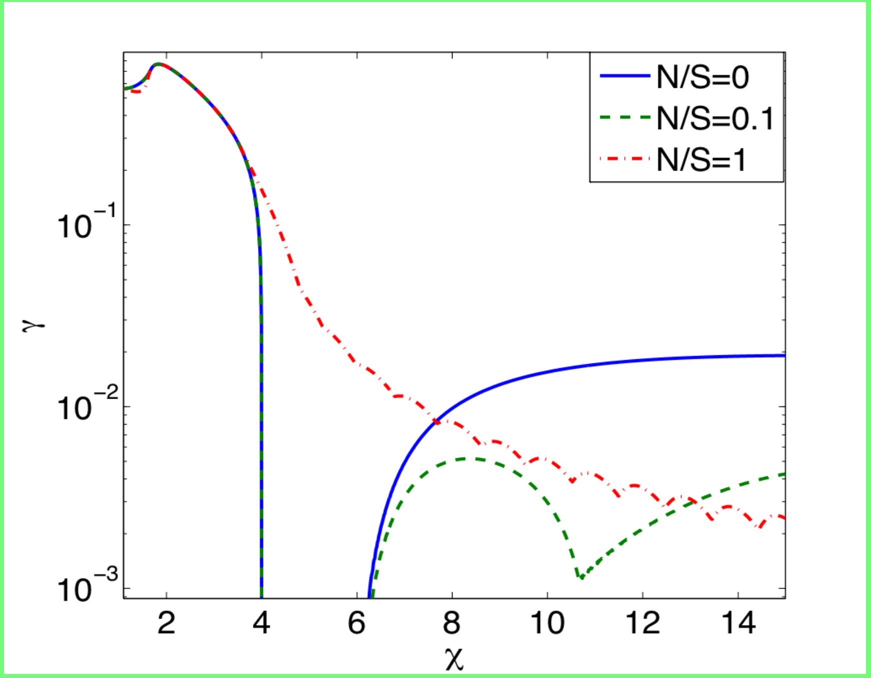
where $\boldsymbol{\xi}$ is the Lagrangian displacement, $\mathbf{a}=\lambda\nabla\psi$, for constant λ is the wavenumber component $\perp$ to $\mathbf{u}$, with ψ being the stream function.
The operator $\mathbf{C}$ is such that

$$\mathbf{C}(\boldsymbol{\xi})=-\frac{\boldsymbol{\xi}}{\rho}\cdot\nabla(\nabla P)-\boldsymbol{\xi}\cdot\nabla(\nabla\Phi)$$

System of ODEs with periodic coefficients – Floquet problem – instability bands associated with resonances between mode periods and streamline period

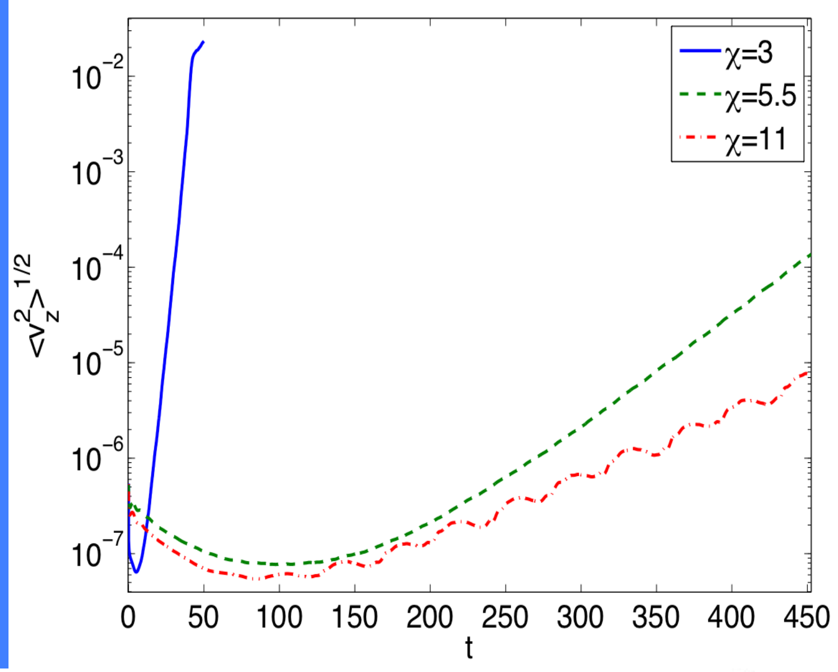
Linear stability of Kida vortex core (barotropic with no stratification)

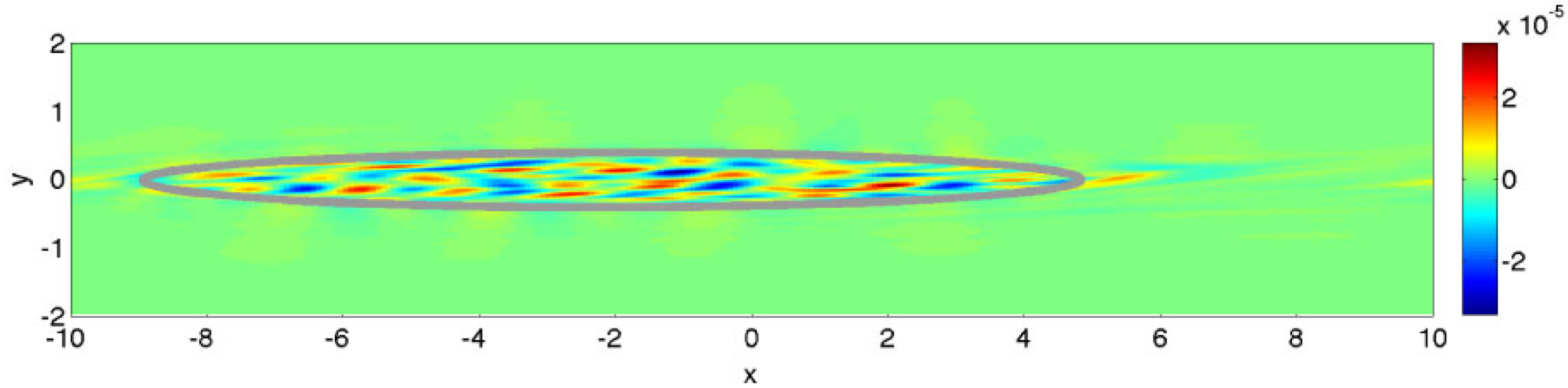




Note string instability for aspect ratios less than 4 + parametric resonant bands….
(Gap between 4 and 6 when core considered)

Slow instablity of large aspect ratio vortex and in the instability gap with simulations





Strong instability of vortices with aspect ratio less than four Is analagous violation of Rayleigh's criterion

$\kappa^2 < 0$

Consider a wide class of local steady flows (u_x^0, u_y^0). The linearized equations of motion are

$$\frac{Dv_i}{Dt} + v_j \frac{\partial u_i^0}{\partial x_j} - 2\Omega \epsilon_{ijz} v_j = -\frac{\partial \Pi'}{\partial x_i}, \qquad *$$

where Π' is the pressure perturbation. The operator

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j^0 \frac{\partial}{\partial x_j}$$

denotes the convective derivative. We now assume the z dependence is through a factor $\exp(ik_z z)$, where k_z is very large. From the incompressibility condition $\nabla \cdot \mathbf{v} = 0$, and the z component of * we conclude that $\Pi' = O(k_z^{-2})$. Thus it may be dropped from the x and y components

$$\frac{Dv_x}{Dt} + v_x \frac{\partial u_x^0}{\partial x} = -v_y \left(\frac{\partial u_x^0}{\partial y} - 2\Omega \right),$$

$$\frac{Dv_y}{Dt} + v_y \frac{\partial u_y^0}{\partial y} = -v_x \left(\frac{\partial u_y^0}{\partial x} + 2\Omega \right).$$

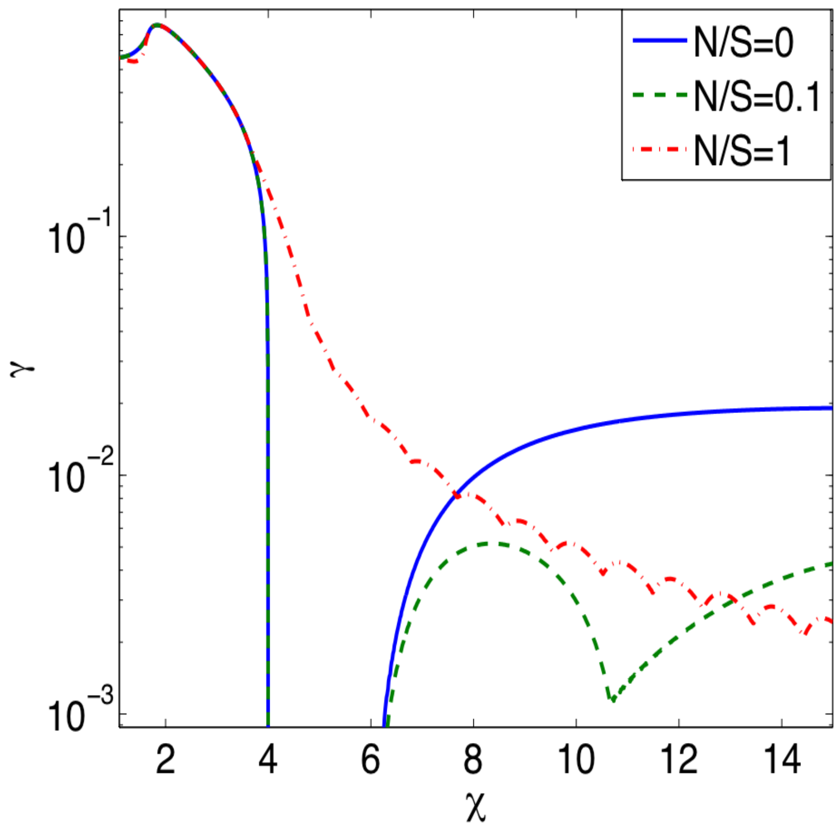
Noting that the time derivative is to be taken on a fixed streamline, we have two first order ODEs with periodic coefficients, the period being the time to circulate round the chosen streamline. For a Kida vortex the coefficients are constant. In the general case one has a Floquet problem on every streamline.

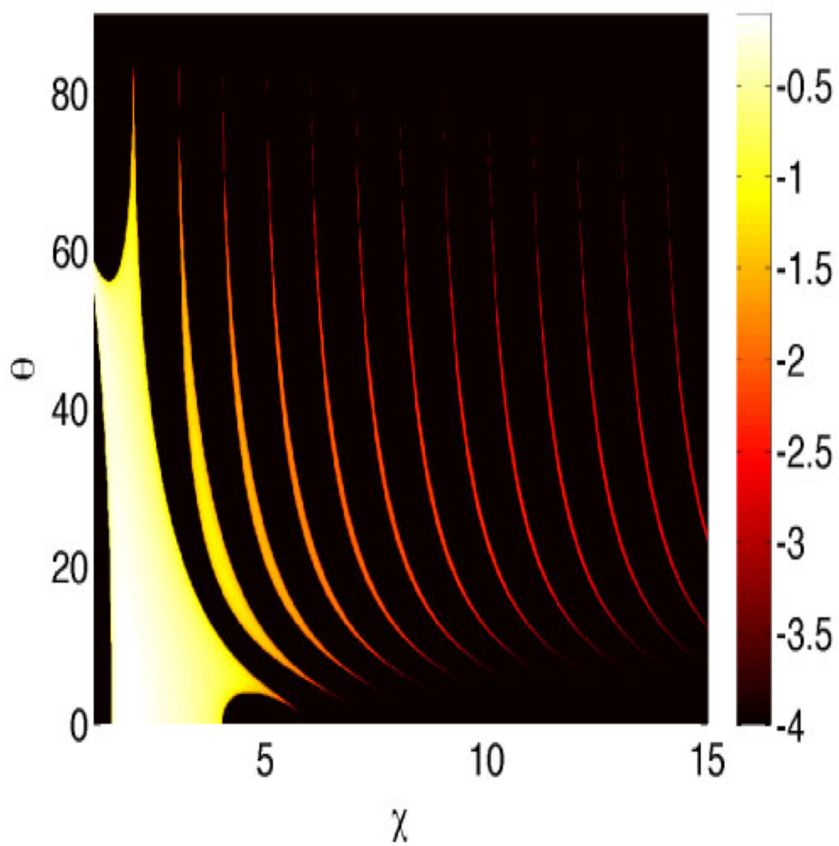
One can prove that if everywhere on a chosen streamline

$$\left(\frac{\partial u_x^0}{\partial y} - 2\Omega \right) \left(\frac{\partial u_y^0}{\partial x} + 2\Omega \right) > 0,$$

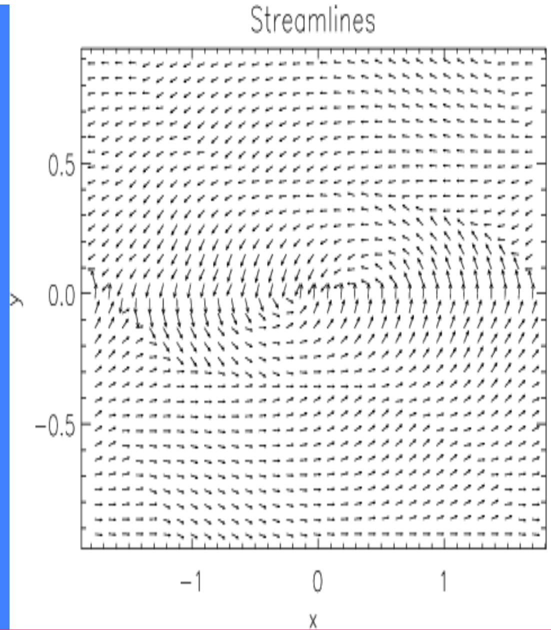
there will be an instability.

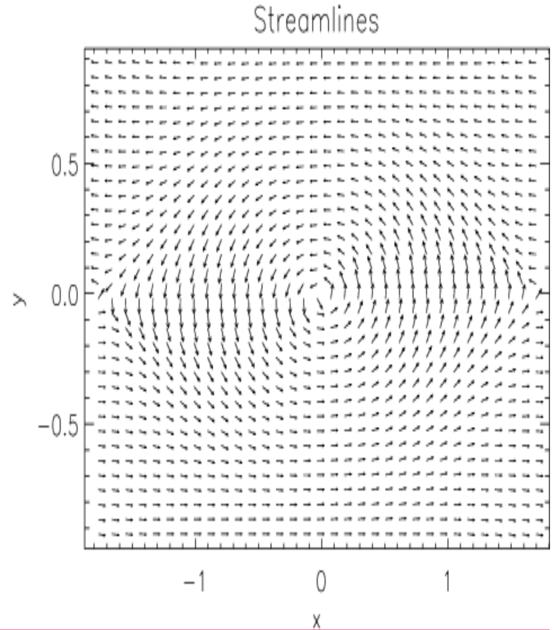
Linear stability of Kida vortex core with restoring forces due to vertical stratification included for perturbations (no gap)



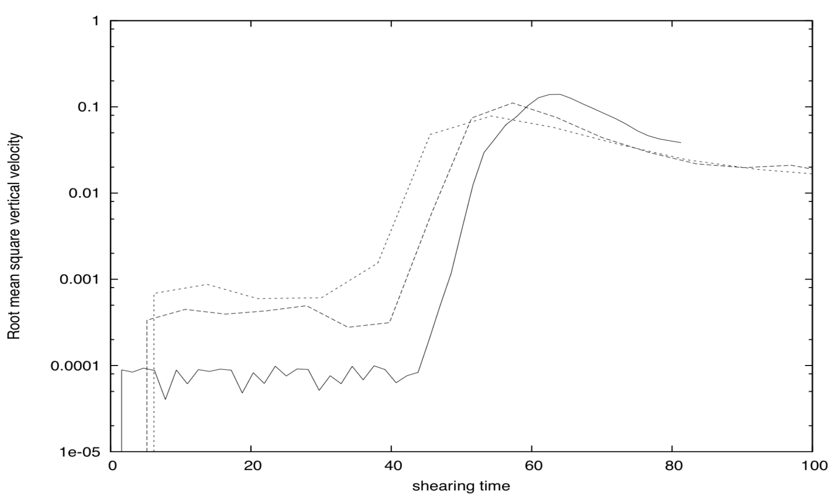


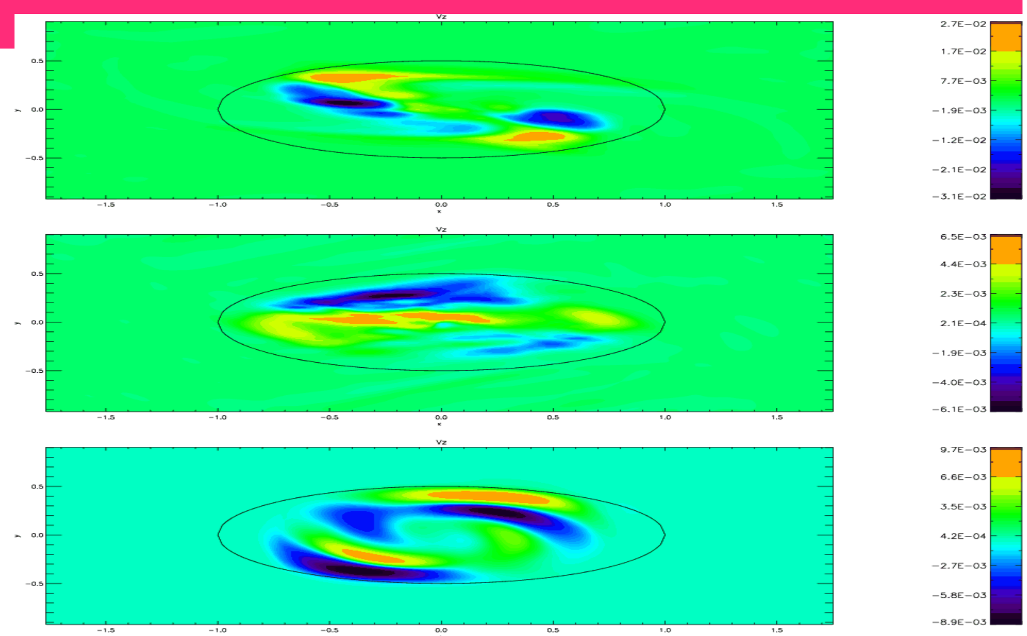
Instability of small aspect ratio vortex. Compressibility included. Isothermal disk with vertical gravity





Δv/c = 0.13, 0.65 & 1.3

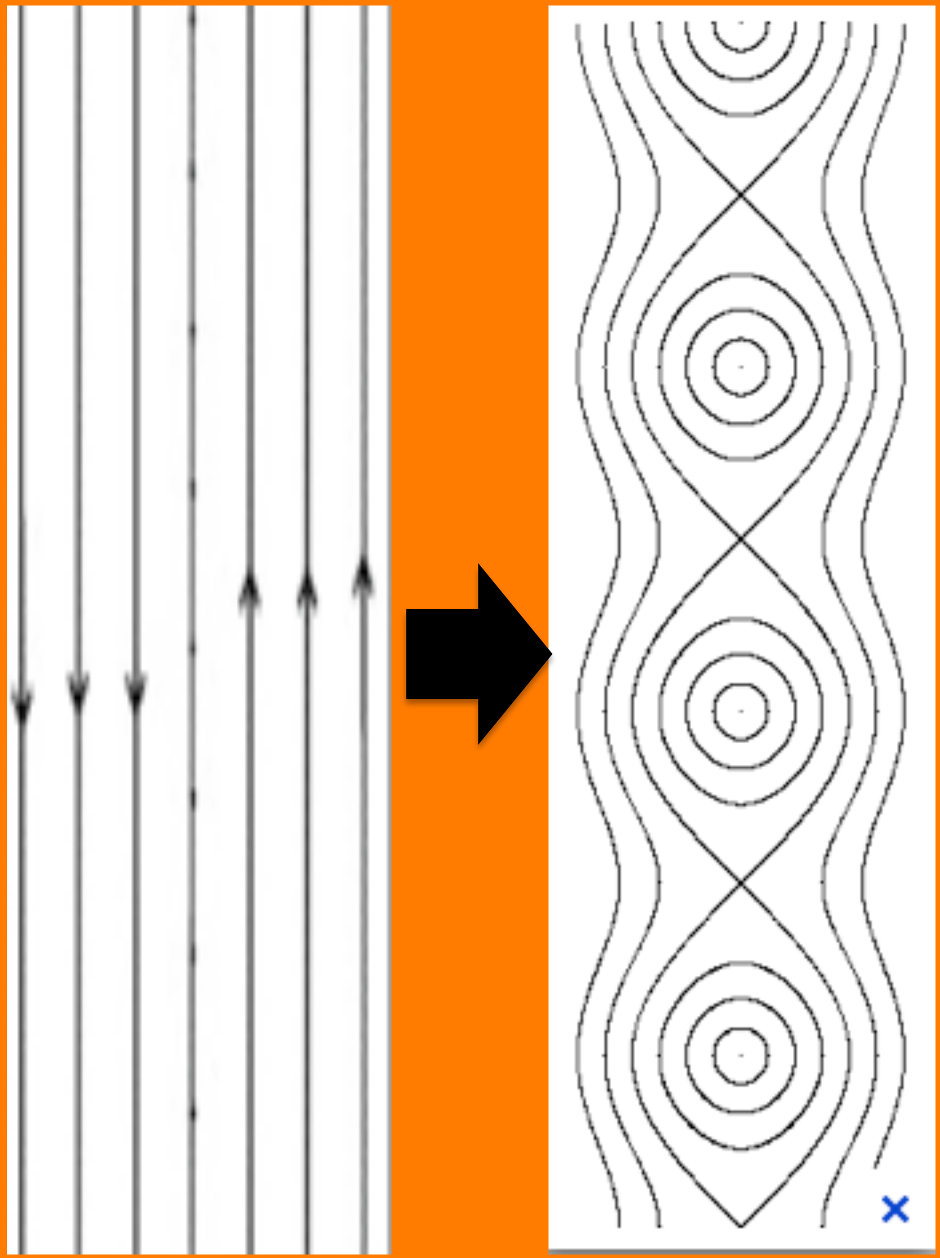




Baroclinic instability mechanism for vortex production: Depends on having heat diffusion/cooling and imposed global P & T gradients. Could work on small scales depending on thermal timescale.

….see eg. Petersen et al (2007), Lesur & Papaloizou (2010), Lyra & Klahr (2011)

For baroclinic instability to work need finite amplitude perturbation. Need vortex structure to start with won't work from linear shear



Two basic equations

The vorticity equation

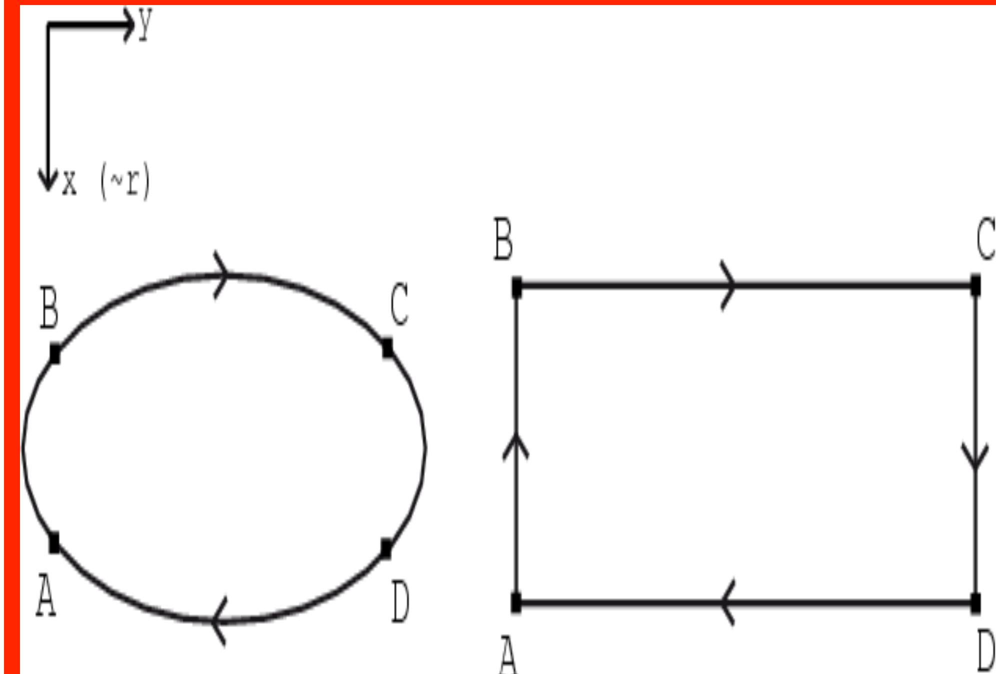
$$\partial_\tau \omega + (\vec{v} \cdot \vec{\nabla} - Sx\partial_y)\omega = \Lambda N^2 \partial_y \theta + \nu \Delta \omega$$

Equation for entropy perturbation

$$\partial_\tau \theta + (\vec{v} \cdot \vec{\nabla} - Sx\partial_y)\theta = v_x/\Lambda + \mu \Delta \theta$$

where $\Lambda = -\frac{d\theta_0}{dx}$ **and** $N^2 = -\frac{1}{\rho C_p}\frac{dP}{dx}\frac{d\theta_0}{dx}$

Driving the baroclinic instability



A streamline in a vortex undergoing the SBI.

A fluid particle is accelerated by buoyancy effects on the A-B and C-D branches enhancing the circulation in the vortex.

Cooling occurs on the B-C and D-A branches.

Equation for the enstrophy is

$\langle \omega^2/2 \rangle = \int \mathrm{d}x\mathrm{d}y\ \omega^2/2$:

$$\partial_t \langle \omega^2/2 \rangle = \Lambda N^2 \langle \omega \partial_y \theta \rangle - \nu \langle |\vec{\nabla} \omega|^2 \rangle.$$

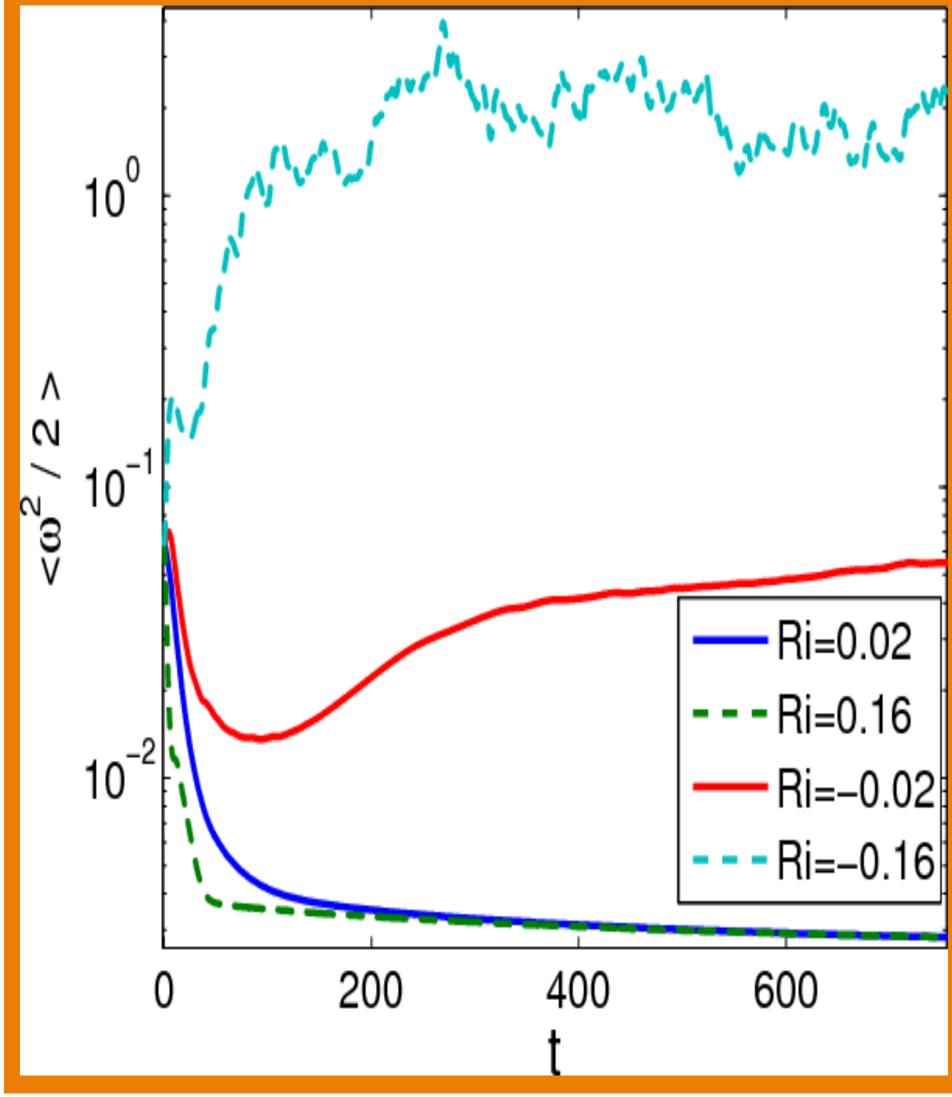
Typical growth rate

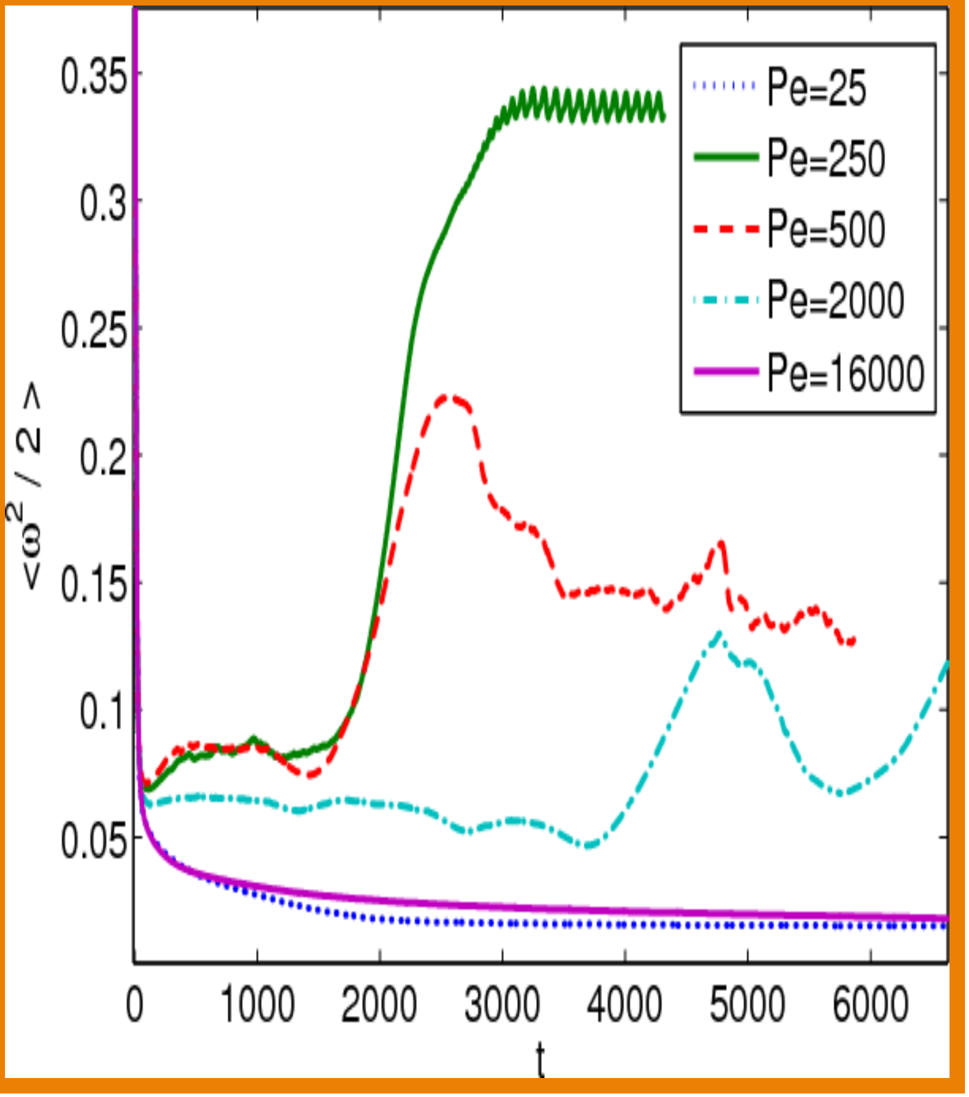
$$\gamma \sim -\mathrm{Ri}\, \Omega^2 \tau_c F(\Omega\tau_c)$$

where $F(0) = 1$ and
$F(x) \rightarrow 0$ for large x

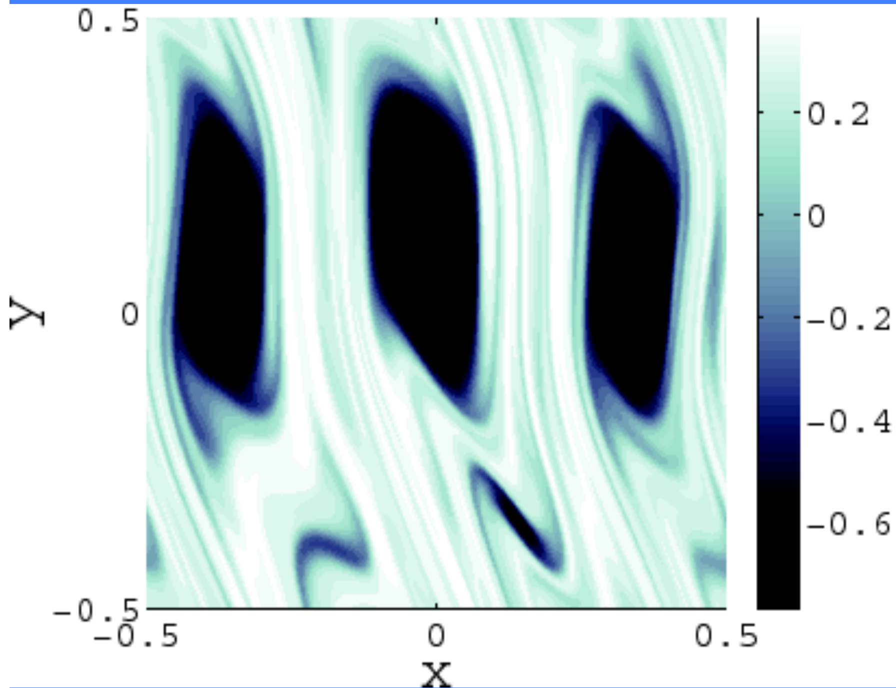
(τ_c = cooling time
Ri = Richardson number = $4N^2/(9\Omega^2)$)

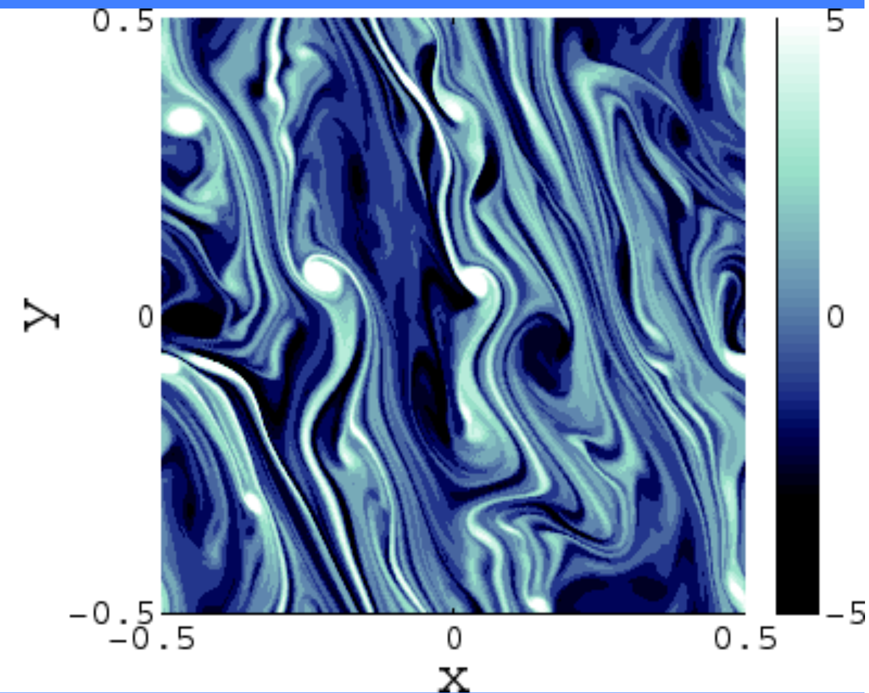
Growth of enstrophy for different Ri and Pe (Boussinesqu)



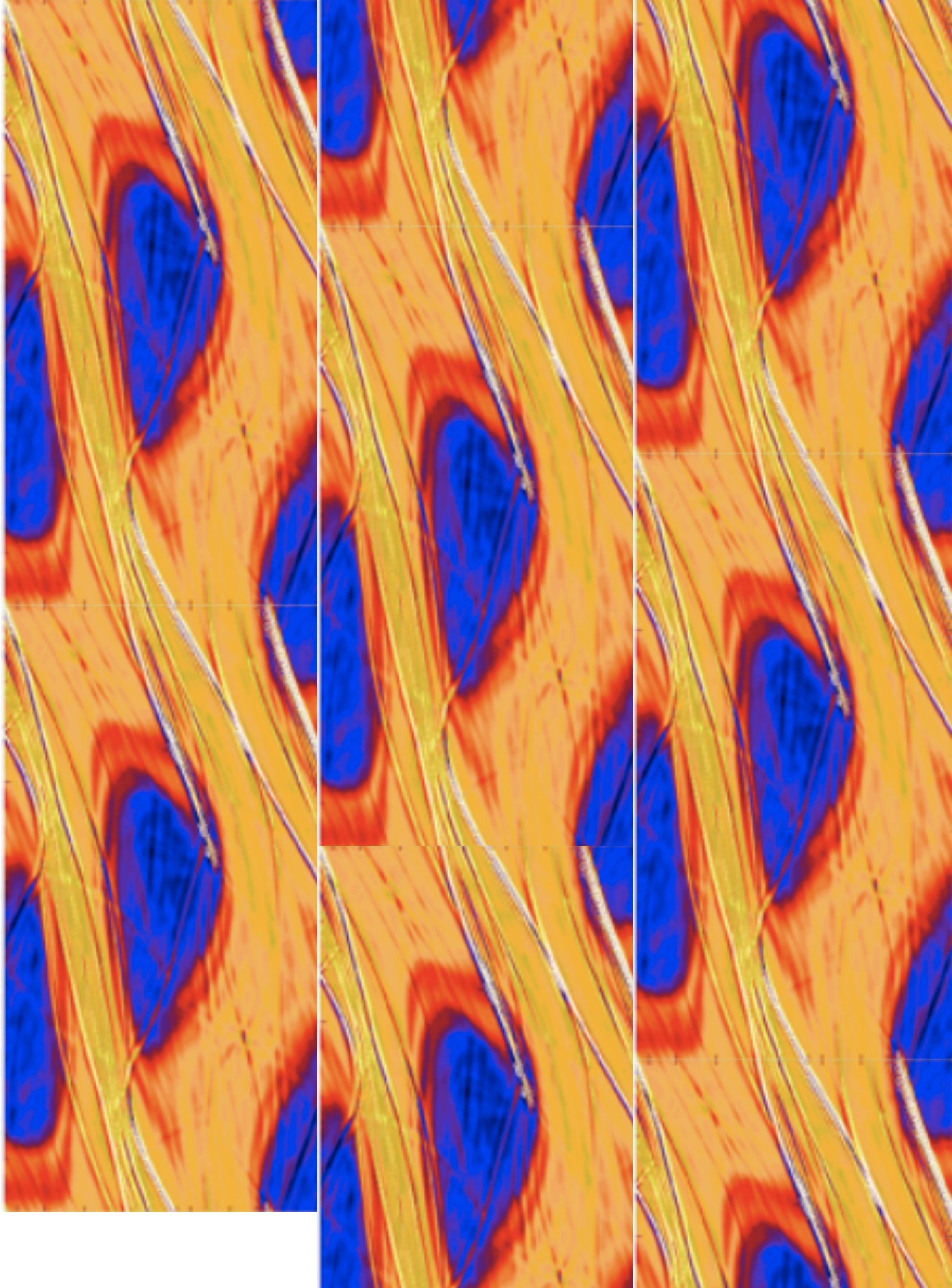


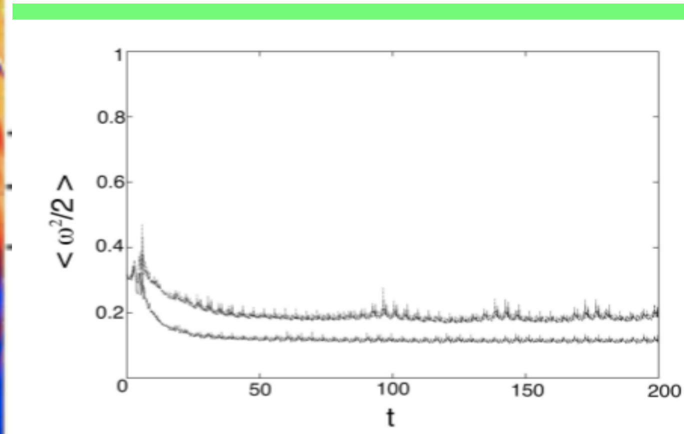
Development baroclinic instability in 2D for two different Richardson



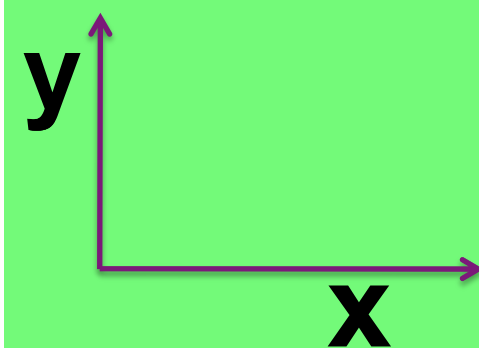


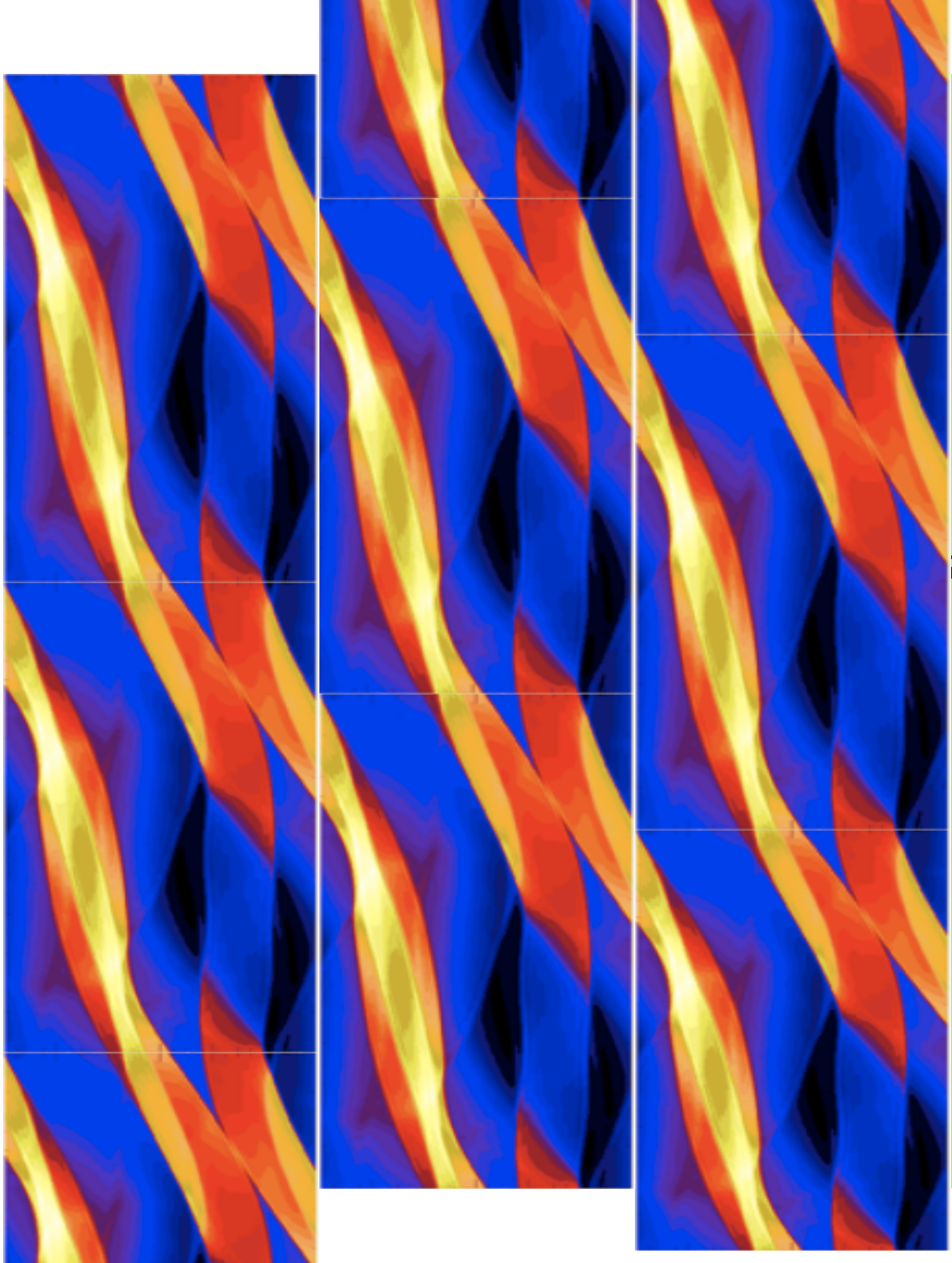
Vorticity map for $Ri = -0.02$ (left) and $Ri = -0.16$ (right) at t=500. Small scales are dominant for larger $|Ri|$.





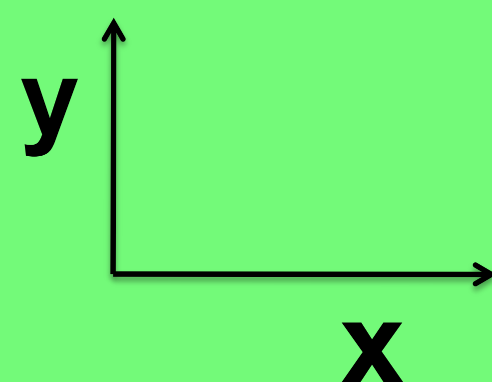
Vorticity map for the compressible case over 3.6Hx7.2H. Imposed heating and thermal diffusion maintains oscillatory T profile with max. Ri ~ -0.7 and Pe ~ 400.



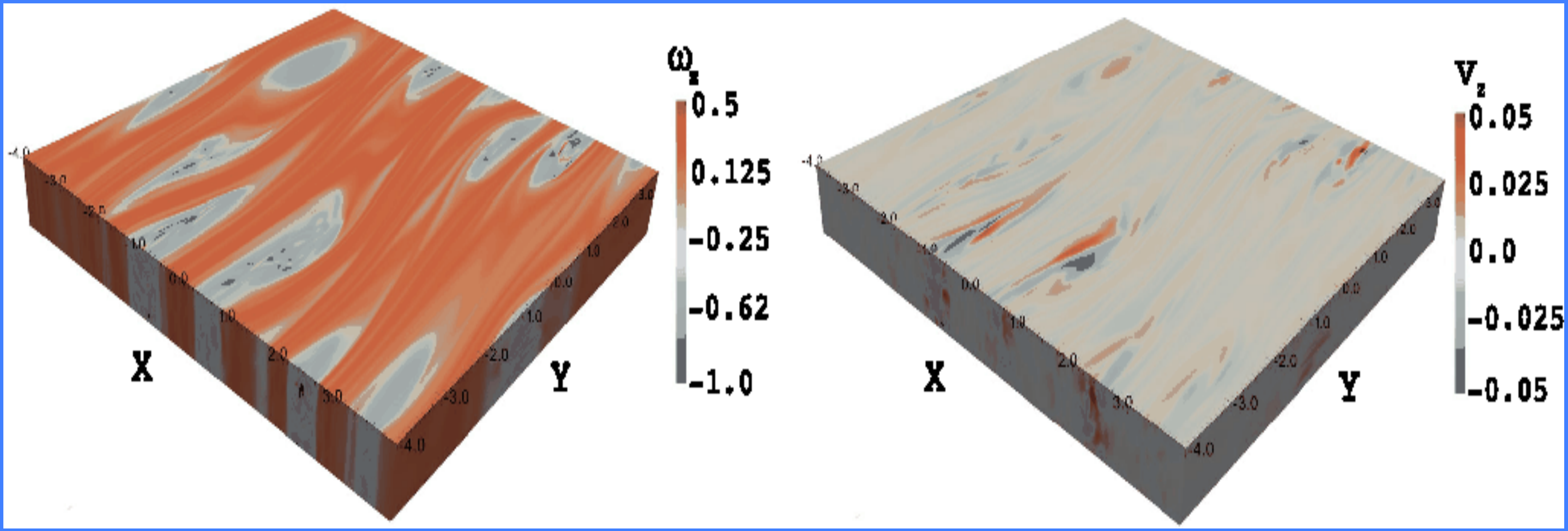


Surface density map for the compressible case

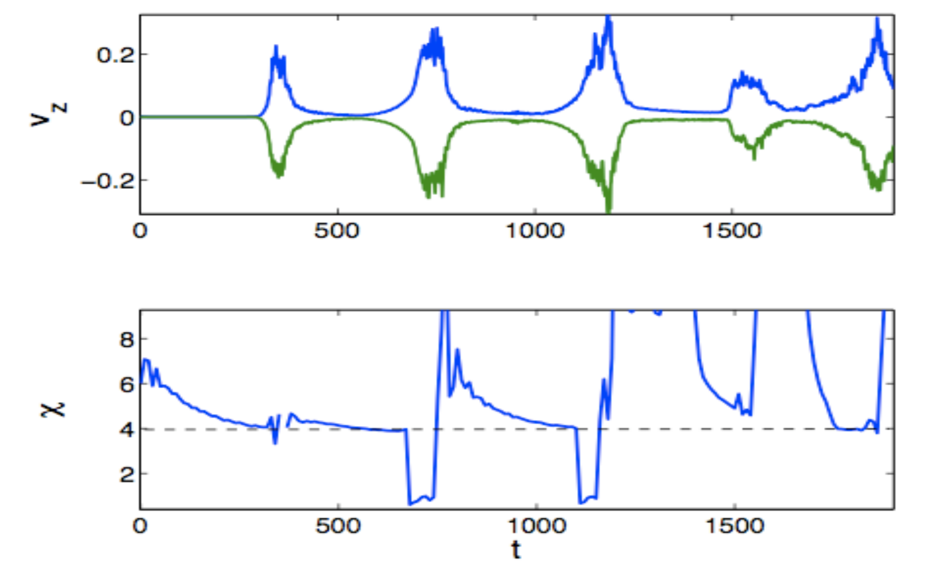
Trailing density waves lead to angular momentum transport with
$\alpha \sim 0.03$



3 D simulations



SBI cyclically drives vortex from large to small aspect ratio - Onset of instability drives vertical motions driving the aspect ratio back to large values



Conclusions

Vortex generating instabilites are robust in non magnetic disks.

They avoid serious instabilities if sufficient elongated.

They may play some role as sites for particle accumulation.
Extent to which this works unclear