The Stability of Dust Laden Vortices

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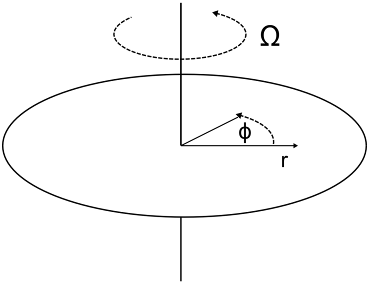
Introduction

- Anticyclonic vortices can form coherent, long-lived structures (e.g Bracco et al. 1999)
- There are many mechanisms to form vortices in discs (Baroclinic, Rossby Wave, MRI, Streaming Instabilities)
- Vortices can trap dust
- BUT does the presence of dust in vortices destabilise it? (Chang and Oishi's heavy-core instability?)

The Equilibrium Model - Setting up the problem I

We start with a disc that is

- Keplerian $\left(\Phi = -\frac{GM}{r},\ \Omega^2 = \frac{GM}{r^3}\right)$
- Inviscid $(\nu = 0)$
- Incompressible $(\nabla \cdot \mathbf{u} = 0)$
- 2D $(\mathbf{u} = \nabla \times \psi \hat{\mathbf{z}})$



Time-independent Navier-Stokes:

$$\nabla \left(\frac{P}{\rho} + \tilde{\Phi} + \frac{1}{2} |\nabla \psi|^2 \right) = - \left(\omega + 2\Omega + \frac{P}{\rho^2} \frac{d\rho}{d\psi} \right) \nabla \psi$$

$$\Rightarrow \ \nabla F = -Q \nabla \psi \qquad \Rightarrow \ \frac{dF}{d\psi} = -Q$$

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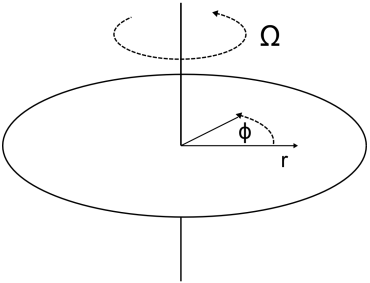

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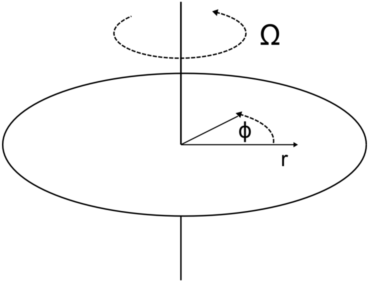

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The Equilibrium Model - Setting up the problem II

Move to the shearing sheet approximation. Set the Keplerian disc as the background flow (F_0, Q_0, ψ_0). Write

$$
\begin{aligned}
\psi &= \psi_0 + \psi_1 \\
&\vdots \\
\Rightarrow \nabla^2\psi_1 &= \frac{dF_1}{d\psi} + \frac{P}{\rho^2}\frac{d\rho}{d\psi} = \mathcal{A}(\psi) + \frac{P}{\rho}\mathcal{B}(\psi)
\end{aligned}
$$

- Prescribe vorticity ($\mathcal{A}$) and density ($\mathcal{B}$) sources
- Start with an initial guess of where the bounding streamline is
- Solve iteratively using the 2D Green's function

$$\psi(\mathbf{r}) = \frac{1}{2\pi}\iint \log|\mathbf{r}-\mathbf{r}'| \left[\mathcal{A}(\psi) + \frac{P}{\rho}\mathcal{B}(\psi)\right] d^2\mathbf{r}'$$

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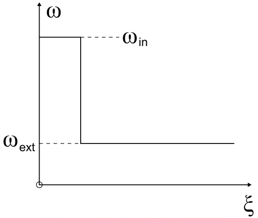
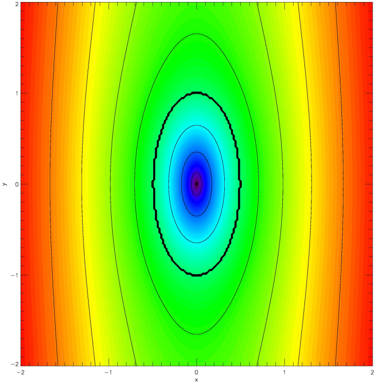
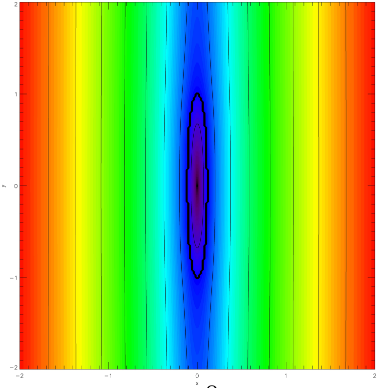
Kida Vortex

- *Top hat* vorticity profile
- Flat density profile
- Elliptical streamlines
- Behaviour dependent on *aspect ratio*, χ.



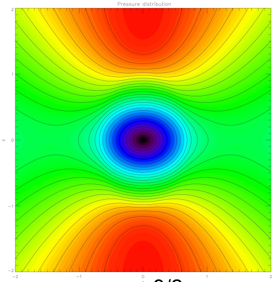


χ=2

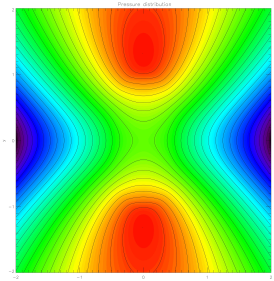


χ=8

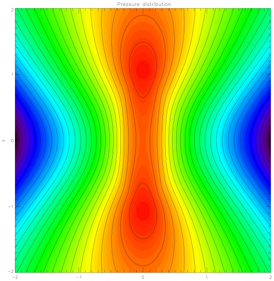
Kida Vortex - Pressure Distributions



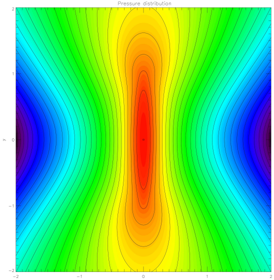
$\chi = 3/2$



$\chi = 2$

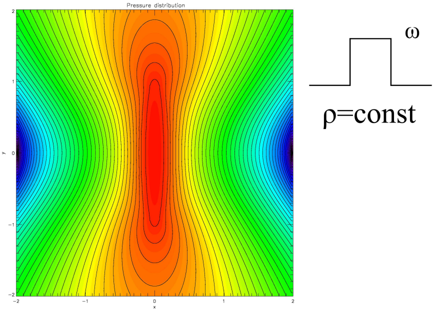


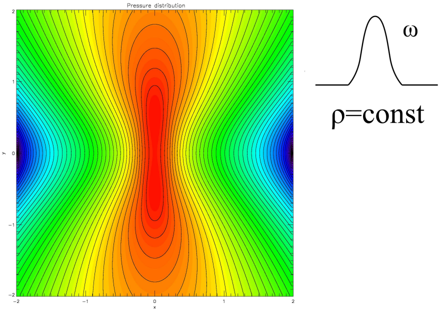
$\chi = 3$

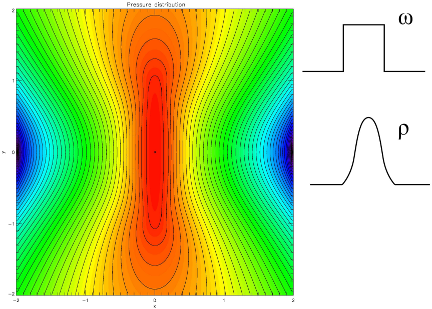


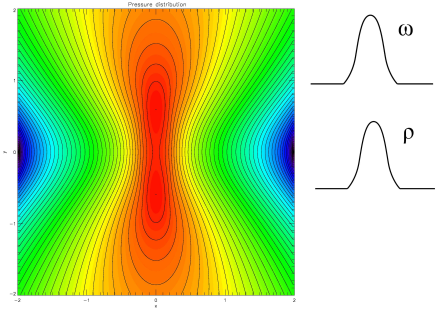
$\chi = 8$

Pressure Distributions - Other Equilibrium Solutions









Stability Calculations

Make the perturbation to calculated equilibrium solutions:

$$\begin{aligned} \mathbf{u} &\longmapsto \mathbf{u_0} + \mathbf{u}' \\ P &\longmapsto P_0 + p' \\ \rho &\longmapsto \rho_0 + \rho' \end{aligned} \tag{1}$$

Perturbing and linearizing the motion and continuity equations:

$$\frac{Du'_x}{Dt} + \mathbf{u}'.\nabla u_{x0} - 2\Omega u'_y = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \frac{\rho'}{\rho_0^2}\frac{\partial P_0}{\partial x},$$

$$\frac{Du'_y}{Dt} + \mathbf{u}'.\nabla u_{y0} + 2\Omega u'_x = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \frac{\rho'}{\rho_0^2}\frac{\partial P_0}{\partial y},$$

$$\frac{Du'_z}{Dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} \qquad \frac{D\rho'}{Dt} = -\mathbf{u}'.\nabla\rho_0$$

Hydrostatic balance $\Rightarrow p' \propto e^{ik_z z}$. Take large k_z limit.

Write as a matrix equation:

$$\mathbf{X}' = \left(u'_x,\ u'_y,\ \rho'\right)^\mathsf{T} \qquad \Rightarrow \frac{D\mathbf{X}'}{Dt} = \mathsf{M}\mathbf{X}'$$

Resulting matrix:

$$\mathsf{M} = \begin{pmatrix} -\partial_{xy}\psi & -\partial_{yy}\psi + 2\Omega & \rho_0^{-2}\partial_x P_0 \\ \partial_{xx}\psi - 2\Omega & \partial_{xy}\psi & \rho_0^{-2}\partial_y P_0 \\ -\partial_x \rho_0 & -\partial_y \rho_0 & 0 \end{pmatrix}$$

- Take numerical equilibrium solution
- Calculate M on each streamline as a function of arclength
- Solve ODE for $\mathbf{X}'$

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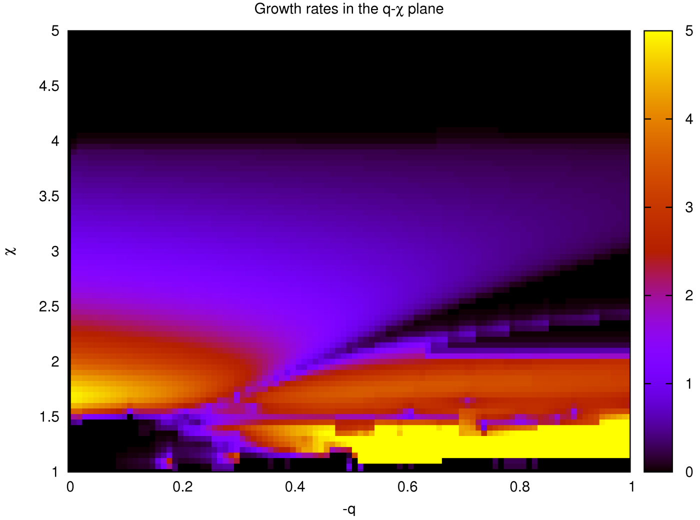
Preliminary Results



q measure of the density perturbation: $\rho = \rho_0 \exp\left\{ q \left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \right) \right\}$.

Summary

- Find equilibrium solutions for a variety of density and vorticity profiles
- Perform linear stability analysis on these solutions (initially $k_z \to \infty$, later general wavelength dependence)
- For the Kida vortex with $\chi > 4$ have stability for wide range of density perturbations in $k_z \to \infty$ case.