The unsteady competition between Rossby and Stratorotational instabilities in PP disks

. . .

and other new results regarding Goldreich-Schubert-Fricke Instability for disks

Orkan (Matt) Umurhan
School of Natural Sciences
University of California, Merced

In Collaboration With: O. Gressel, R. Nelson & R. Yellin-Bergovoy

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Outline

- Part I: Rossby and Gravity Mode Instabilities for disks
- Part II: Goldreich Schubert Fricke Instability for Disks
- Part III: Amalgamation

Previous Discussions

1. Rossby Wave Instability (RWI) - barotropic process
   Instability of at least 2 Rossby waves in shear separated by evanescent zone
   - Fluid Dynamics Historical (Rayleigh 1880, limiting form → Kelvin-Helmholtz Instability)
   - Meteorological (Bretherton 1965 in terms of "potential vorticity", Hoskins et al. 1985, · · ·),
   - Astrophysical (Lovelace, Li et al. 2000, Meheut 2010, 2012, Varnier & Tagger, and others)
   - Interpretation as "CRW's" (Baines & Mitsudera 1994, Heifetz et al. 1999, Umurhan 2010)

2. Gravity Wave Instability (GWI) -
   resonant interactions with shear flow
   - Meteorological (Satomura 1982, Knessl & Keller 1994, Ford 1994, Balmforth 1999)
   - Laboratory Taylor-Couette (Yavneh 2000, Le Dizes 2009,...)
   - Fluid dynamics general - aka "Radiative Instability" (Schecter & Montgomery 2010, Le Dizes Group 2008-Present)
   - Astrophysical (Dubrulle 2004 as "Stratorotational Instability" **SRI**, Le Dizes Group 2008-Present)

Non-Asymptotic Reduction of Disk Equations

1. Meheut et al. numerical analysis:
   - isentropic equation of state $P = \rho^\gamma$
   - disk midplane symmetric disturbances
2. Minimal Assumptions Assume following central star potential gradients (very good for thin/cold disks)

$$\frac{\partial \phi}{\partial r} \approx \Omega_0^2 r_0 (r_0/r)^2 \;\; \& \;\; \frac{\partial \phi}{\partial z} \approx -\Omega_0^2 z (r_0/r)^3$$

3. Expansions for radial (u) and azimuthal (v) and enthalpy (Π),

$$\left\{ \begin{array}{c} u(x,y,z,t) \\ v(x,y,z,t) \\ \Pi(x,y,z,t) \end{array} \right\} = \left\{ \begin{array}{c} u_0(x,y,t) \\ v_0(x,y,t) \\ \frac{1}{2}\Omega_0^2 (h_0(x,y,t)^2 - z^2) \end{array} \right\} + \sum_{k=1}^{\infty} \left\{ \begin{array}{c} u_{2k}(x,y,t) \\ v_{2k}(x,y,t) \\ \Pi_{2k}(x,y,t) \end{array} \right\} z^{2k}$$

and vertical velocities (w)

$$w(x,y,z,t) = \Omega_1(x,y,t) z + \sum_{k=1}^{\infty} \Omega_{1+2k}(x,y,t) z^{1+2k}$$

4 Truncation procedure similar to other studies see recent e.g. Umurhan 2008, Lin 2011,2012: $n = 0$ truncation only small box shown here - carries over to global disk as well

2D shallow H_2O eqns with Kep. shear

$$\begin{aligned}
\frac{du_0}{dt} - 2\Omega_0 v_0 &= -\partial_x \tfrac{1}{2}\Omega_0^2 h_0^2 \\
\frac{du_0}{dt} - \tfrac{1}{2}\Omega_0 u_0 &= -\partial_y \tfrac{1}{2}\Omega_0^2 h_0^2 \\
\frac{dh_0}{dt} &= \Omega_1 h_0 \\
\partial_x u_0 + \partial_y v_0 &= -\frac{\gamma+1}{\gamma-1}\Omega_1 .
\end{aligned}$$

$$\frac{d}{dt} \equiv \partial_t + u_0\partial_x + (v_0 - q_{\rm sh}\Omega_0 x)\partial_y$$

Comments

- shear/Rossby waves
- gravity waves
- no acoustics
- strong shear

Formulation in terms of Potential-Vorticity/Divergence/Enthalpy

$$\zeta \equiv \partial_x v - \partial_y u \text{ (Vorticity)}, \qquad \mathcal{D} \equiv \partial_x u + \partial_y v \text{ (Divergence)}$$

$$\Pi \equiv \tfrac{1}{2} h^2 \text{ (Enthalpy)}, \qquad \mathcal{Z} \equiv \frac{\Omega_0(2-q)+\zeta}{\Pi^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}} \text{ (Potential Vorticity)}$$

Evolution Equations

$$\begin{aligned}
\frac{d\mathcal{Z}}{dt} &= 0 \\
\frac{d\Pi}{dt} &= -2\left(\frac{\gamma-1}{\gamma+1}\right)\Pi\mathcal{D} \\
\left(\partial_t - q_{\rm sh}\Omega_0 x \partial_y\right)\mathcal{D} &= -\nabla^2\left[\Pi + \tfrac{1}{2}\left(u^2+v^2\right)\right] + 2\Omega_0 q_{\rm sh}\partial_y u \\
&\quad + \partial_x v(\zeta + 2\Omega_0) - \partial_y u(\zeta + 2\Omega_0)
\end{aligned}$$

with diagnostic relations :

$$u = -\partial_y\psi + \partial_x\phi, \qquad v = \partial_x\psi + \partial_y\phi, \quad \mathcal{D} = \nabla^2\phi, \quad \zeta = \nabla^2\psi.$$

Linear Theory - in a shear layer

- Perturbations of uniform shear state between walls (ala classic SRI)

  (• Theoretical results carry over when walls are replaced with jumps in mean Potential Vorticity)

- Consider uniform state $h = h_0 =$ constant and $\Pi' = \hat{\Pi}(x) \exp ik(y - ct)$ + c.c.
- Parameters:

  (wall half-width) Δ, (shear rate) q_{sh}, (wavenumber) $k > 0$, (grv. wavespeed) $c_g^2 \equiv \frac{1}{2}\Omega_0^2 h_0^2 \frac{\gamma - 1}{\gamma + 1}$,

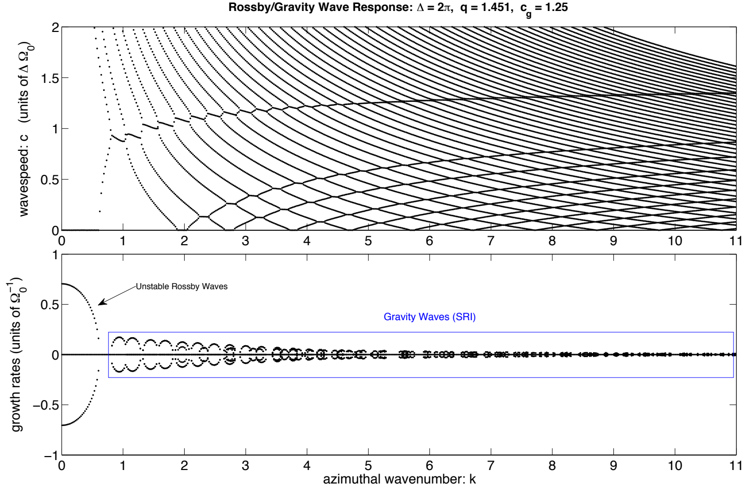
perturbation in enthalpy - find complex c

$$\frac{k^2}{c_g^2}(c - q_{\text{sh}}\Omega_0 x)^2 \Pi' = -\left(\partial_x^2 - k^2 - \frac{\kappa^2}{c_g^2}\right)\hat{\Pi} + 2\Omega_0 ikq_{\text{sh}}\hat{u}$$

$$\hat{u}(x) = -\frac{ik}{2c_g^2}\int_{-\Delta}^{\Delta} e^{-k|x-x'|}\left[\text{sgn}(x - x')(c - q_{\text{sh}}\Omega_0 x') + \frac{1}{k}\Omega_0(2 - q_{\text{sh}})\right]\Pi(x')dx'$$

with $\kappa^2 \equiv 2(2 - q_{\text{sh}})$

Linear Response - layer



Linear theory - free shear analysis of gravity waves easier!!

- Sheared coordinate frame transformation, $\Pi'(t)e^{i\ell(t)x+iky}$ $\quad \ell(t) = \ell + q_{sh}\Omega_0 kt$

initial value ODE

$$\left[\partial_t^2 + \kappa^2 + c_g^2(k^2 + \ell(t)^2) + \frac{2q_{sh}\Omega_0 k}{k^2+\ell(t)^2}\left(\ell(t)\partial_t + k(2-q_{sh})\right)\right]\Pi' = 0$$

Solutions for $q_{sh}\Omega_0 kt \gg 1$

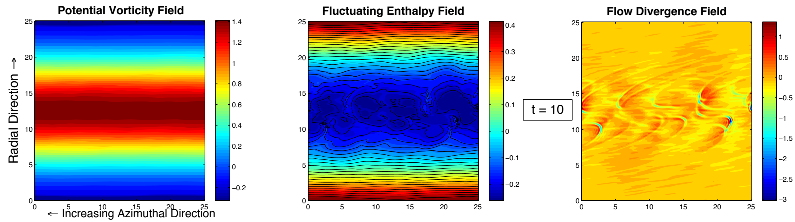
$$\Pi' = \sqrt{k^2+\ell(t)^2}\,\mathrm{D}\left[(1+i)t\right], \quad \mathrm{D}\left[(1+i)t\right] \leftrightarrow \text{Parabolic Cylinder Fcn}$$

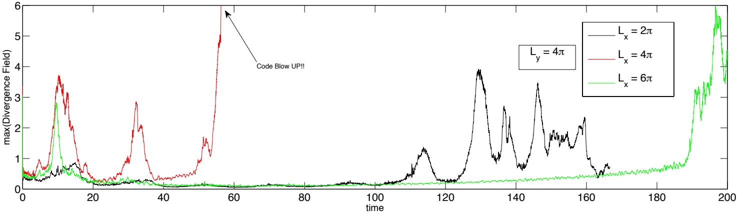
asymptotic forms for large t

$$\Pi' \longrightarrow \text{const. amp. oscillations}, \qquad \mathcal{D}' \longrightarrow \underline{q_{sh}t} \times \left(\text{const. amp. oscillations}\right)$$

GWI - runs: no RWI operating

flow fields

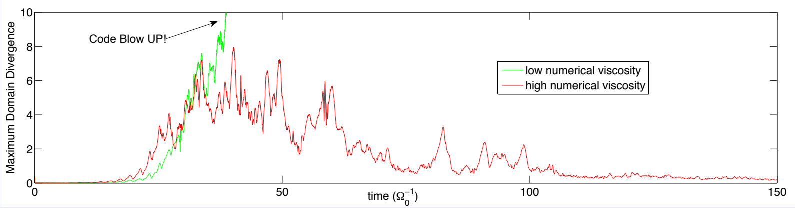




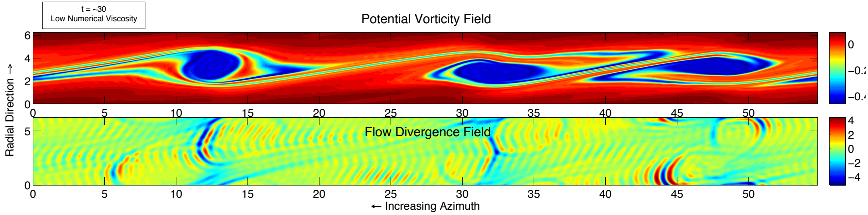
RWI vs. GWI Runs - Preliminary Indications

- Hypothesis to test: Do quasigeostrophic results carry over to SWT?...yes, sort of.
- Pseudospectral tests 256×1024 runs with ∇^{16} viscosity.

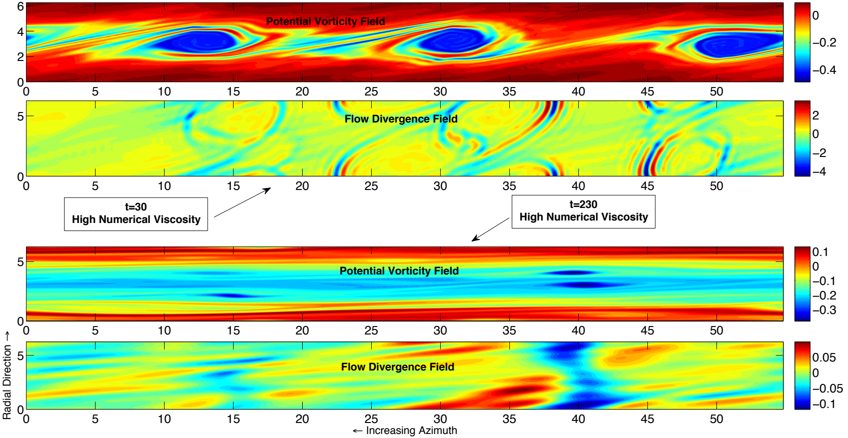
Current Status: Need high numerical viscosity to control gravity waves



Flow Fields - low viscosity



Flow Fields - high viscosity



GSF Instability - unstable inertial waves - (nearly incompressible disturbances)

1. Goldreich-Schubert-Fricke (1967/68) [Also Urpin 2003, Arlt & Urpin (2004)]

   – mean rotation not constant on cylinders $\longrightarrow$ $j^2 = R^2\Omega(R,Z)$

   $$\textbf{instability for: } \frac{\partial j}{\partial R} - \frac{\ell_Z}{\ell_R}\frac{\partial j}{\partial Z} < 0 \quad \text{(Solberg-Hoiland)}$$

2. For a "locally isothermal" disk:

   $$T = T_0\left(\frac{R}{R_0}\right)^q \qquad \Longrightarrow \qquad \overline{V}(R,Z) = \overline{V}_{\text{Kep}}(R)\left(1 + \underline{q\left[\frac{H_0^2}{R_0^2}\right]Z^2}\cdots\right)$$

3. cold disks: scale height $\ell_Z = H_0 \ll R_0$ implies (for reference Ω_0 at R_0)

   $$\text{(growth rates)} \sim q\Omega_0\frac{H_0}{R_0} \quad \Longleftrightarrow \quad \text{(on radial disturb. length scales) } \ell_R \sim \frac{H_0}{R_0}H_0.$$

   $$\boxed{\text{For } H_0/R_0 = 0.05 \Longrightarrow \ \ell_R \sim 0.01R_0}$$

Method/Parameters/Results

Model Equations

$$\partial_t \rho + \nabla \cdot \rho \mathbf{u} = 0,$$
$$\partial_t \rho \mathbf{u} + \nabla \cdot \rho \mathbf{u}\mathbf{u} = -\nabla P - \rho \nabla \Phi,$$
$$\partial_t T + \mathbf{u} \cdot \nabla T = -(T - T_{\text{ref}})/\tau_{\text{relax}}$$

with $P = \rho T$ and $\Phi = -GM/R$
or a proper energy equation

$$\partial_t e + \nabla \cdot e\mathbf{u} = -P\nabla \cdot \mathbf{u} + \mathcal{Q}$$

Code and Setup

- Nirvana and Nirvana-III code
  (spherical coordinates) ($N_r = 1300$ and $N_\theta = 1000$)
- Axisymmetric disturbances -
  $r_{\text{in}}/R_0 = 1 \;\; r_{\text{out}}/R_0 = 2$, $Z_{\text{max}}/H_0 = 5$.
- Outflow or reflecting conditions -
  (no observed difference in results)
- q = -1 (constant $H_0/R_0 = 0.05$ over domain)
- seed with random field in KE

Result

Strong Activity when $\tau_{\text{relax}} \to 0$ and $q \neq 0$.

vertical velocity frames

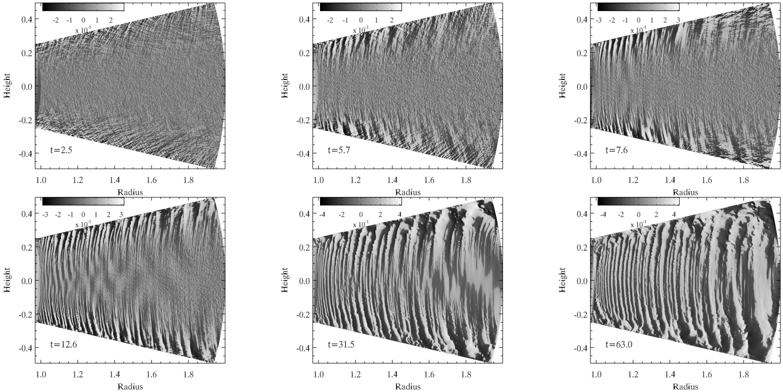


Figure 3. Edge-on contours of the perturbed vertical velocity as a function of R, Z and time for model TR1–0. Note that for clarity, the grey-scale of the image has been streteched by plotting the quantity $\mathrm{sign}(v_Z) \times |v_Z|^{1/4}$. Note that the spectrum bar shows values of $v_Z^{1/4}$.

- Radial Wavelength of dominant growing mode $\sim 0.009 R_0$.

features and clues

component KE

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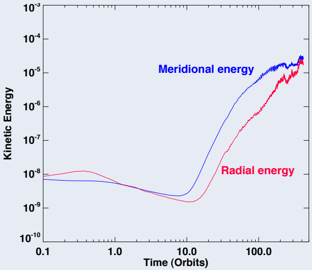


Figure 1. Time evolution of the normalised perturbed kinetic energy in the meridional and radial coordinate directions for model T1R-0 with $p = -1.5$, $q = -1$ and reflecting boundary conditions at the meridional boundaries.

growth rate ~ 0.24 orbit^{-1}

Variation of $\tau_{\rm relax}$

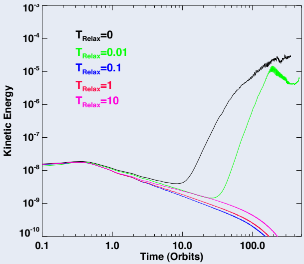


Figure 10. Time evolution of the sum of the (normalised) perturbed radial and meridional kinetic energy in discs where the temperature was initially constant on cylinders, as a function of the thermal relaxation time. Note that only the $\tau_{\rm Relax} = 0$ and 0.01 cases show growth.

- Important Clue: radial velocities dwarfed by vertical velocities

Stripped down model exposing instability

numerically guided asymptotic analysis

- time$^{-1} \sim \Omega_0 H_0/R_0$, $\ell_R \sim (H_0/R_0)^2 R_0$, $\ell_z \sim H_0$, and $v_R \ll v_z$, for $\tau_{\text{relax}} \to 0$ and $q \neq 0$

Reduced equations

examined around fiducial radius $R = R_0$ where $T = T_0 \iff c_{s0}^2$

$$
\begin{aligned}
-2\Omega_0 v &= -c_{s0}^2 \partial_r \ln\rho && \text{Radial Geostrophy!!} \\
\frac{dv}{dt} + \tfrac{1}{2}\Omega_0 u + \frac{\partial \bar{V}}{\partial z} w &= 0 && \text{Azimuthal Mom} \\
\frac{dw}{dt} &= -c_{s0}^2 \partial_z \ln\rho && \text{Vertical Mom} \\
\frac{\partial \tilde{\rho} u}{\partial r} + \frac{\partial \tilde{\rho} w}{\partial z} &= 0 && \text{Anelastic Eqn.!!}
\end{aligned}
$$

with $\tilde{\rho} = \exp(-z^2/2H_0^2)$.

linear theory and perturbation analysis

linear perturbations $\rho \to \rho_0 + \rho'$

Inseparable equation in r and z!!

$$\frac{\partial^2}{\partial r^2}\frac{\partial^2 \rho'}{\partial t^2} + \frac{\partial^2 \rho'}{\partial z^2} = \left(1 + q\frac{\partial}{\partial r}\right) z \frac{\partial \rho'}{\partial z}$$

double instability !! (kq > 1)

solution modes of the inseparable "form":

$$\rho' = \rho(m,k) \sim \sum_{j=1}^{m} e^{st + jr/s^2} \cos kr\, \mathcal{H}_j(z)$$

where m are integer indices > 0:

$$s^2 = (m/k^2)\left(-1 \pm \sqrt{1 - q^2k^2}\right)$$

maximal growth rates (our sims: q=-1, $H_0/R_0 = 0.05$)

$$\text{radial scale of max growth :} \quad \ell_r = \pi|q|\left(\frac{H_0}{R_0}\right)^2 R_0 \Longrightarrow 0.008 R_0$$

$$\text{growth rate of max growth in KE :} \quad s_{\max}^{(\mathrm{KE})} = 2s_{\max} = \sqrt{2m}\pi|q|\left(\frac{H_0}{R_0}\right)\text{orbit}^{-1}$$

$$\Longrightarrow 0.22\sqrt{m}\ \text{orbit}^{-1}$$

Nonaxisymmetric response - Baroclinic RWI driven by this mechanism?

theoretic considerations:

- Axisymmetric instability $\Longrightarrow$ geostrophic mod. of base shear flow

$$2\Omega_0 v = \frac{\partial \ln \rho}{\partial r}$$

- Set up conditions for <u>barotropic RWI and baroclinic extensions</u>

Corresponding Asymptotic Eqns. for further linear/nonlinear analysis:

$$-2\Omega_0 v = -c_{s0}^2 \partial_r \ln \rho$$

$$\frac{dv}{dt} + \frac{1}{2}\Omega_0 u + w\partial_z \overline{V} = -c_{s0}^2 r_0^{-1} \partial_\phi \ln \rho$$

$$\frac{dw}{dt} = -c_{s0}^2 \partial_z \ln \rho$$

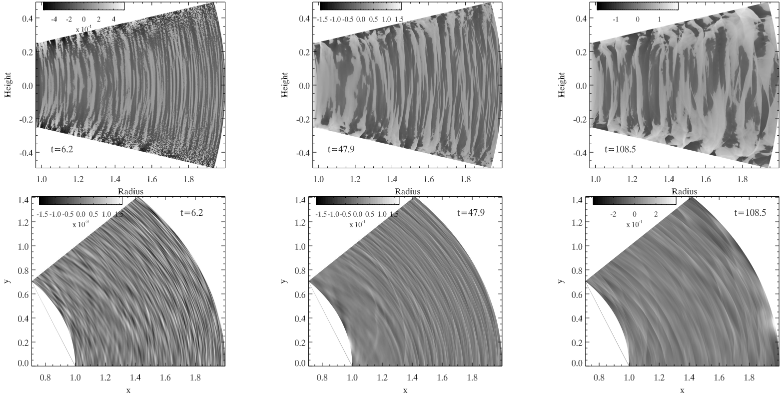
$$\frac{\partial \tilde{\rho} u}{\partial r} + \frac{1}{r_0}\frac{\partial \tilde{\rho} v}{\partial \phi} + \frac{\partial \tilde{\rho} w}{\partial z} = 0$$

$$\frac{d}{dt} = \partial_t + u\partial_x + (v - q\Omega_0(r - r_0))\partial_\phi + w\partial_z$$

with $\tilde{\rho} = \exp(-z^2/2H_0^2)$.

Non-axisymetric **full** numerical runs:

Vertical shear instability in discs 13



- vortex production in-plane $\Longrightarrow$
  Outward angular momentum Transportage!

Nonaxisymmetric response - Baroclinic RWI driven by this mechanism?

Transport Properties

- Effective "α"

$$\sim 2 \times 10^{-3}$$

- Distribution around midplane

Next Stages:

- Sort out linear theory of competing instabilities
- Further examine the requirement that thermal relaxation times must be short!
- Verification via other means

Transport Map

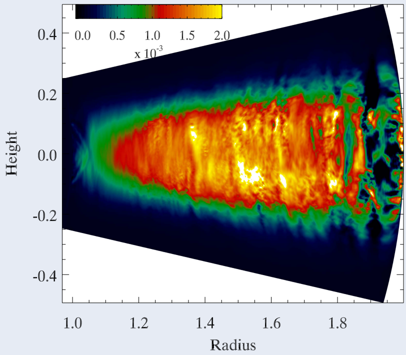


Figure 14. Spatial distribution of the time and horizontally averaged Reynolds stress (normalised by the mean pressure at each radius) for the model T1R–0–3D.